# 有限元方法（FEM）全课程问答手册

**Complete Bilingual Q&A Guide · 176 Topics · L1–L15**

English Answers + 中文解析 · 闭卷考试格式

## 使用说明

- 每题包含：**English Answer**（考试作答用）+ **中文解析**（辅助理解）
- 标记 ⭐ = 样卷已考，🔴 = 高概率未考，🟡 = 中等概率，🟢 = 基础
- 公式统一用 LaTeX 渲染

## L1 — Introduction（FEM基本思想与流程）

### Q1 ⭐ FEM的核心思想是什么?

**EN:** The Finite Element Method (FEM) is a numerical method for solving differential equations by **discretizing a continuous body into a finite number of simple elements**, connected at **nodes**, with the displacement at nodes being the primary unknowns. This converts an infinite-DOF continuous problem into a finite-DOF algebraic system: $\mathbf{KU} = \mathbf{F}$, where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{U}$ is the nodal displacement vector, and $\mathbf{F}$ is the nodal force vector.

**中文:** 有限元方法是将连续体离散为有限个简单单元，通过节点连接，以节点位移为基本未知量，将无限自由度问题转化为有限自由度代数方程组 $\mathbf{KU} = \mathbf{F}$ 的数值方法。K是总刚度矩阵，U是节点位移向量，F是节点力向量。

### Q2 ⭐ 写出FEM分析的8个通用步骤

**EN:** The correct order is:

1. **Discretize** and select element type
2. **Select a displacement function** (shape functions) within each element
3. **Define strain/displacement and stress/strain relationships** ($\boldsymbol{\varepsilon} = \mathbf{Bd}$, $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$)
4. **Derive** the element stiffness matrix and equations ($\mathbf{k}_e = \int \mathbf{B}^T\mathbf{DB}\, dV$)
5. **Assemble** element equations to obtain global equations and **apply boundary conditions**
6. **Solve** for unknown nodal displacements ($\mathbf{K}_{ff}\mathbf{U}_f = \mathbf{F}_f - \mathbf{K}_{fc}\mathbf{U}_c$)
7. **Solve for strains and stresses** ($\boldsymbol{\varepsilon} = \mathbf{Bd}$, $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$)
8. **Interpret** and evaluate the results

**中文:** (1)离散化并选择单元类型→(2)选择位移函数→(3)定义应变/位移和应力/应变关系→(4)推导单元刚度矩阵→(5)装配全局方程并施加边界条件→(6)求解节点位移→(7)计算应变和应力→(8)解释和评估结果。口诀：离-选-定-导-装-求-算-评。

### Q3 ⭐ $\mathbf{KU} = \mathbf{F}$ 中各符号的含义

**EN:** $\mathbf{K}$ = global stiffness matrix (assembled from element stiffness matrices); $\mathbf{U}$ = global nodal displacement vector (the primary unknowns); $\mathbf{F}$ = global nodal force vector (applied forces + equivalent nodal forces).

**中文:** K = 总刚度矩阵（由单元刚度矩阵装配而成）；U = 总节点位移向量（基本未知量）；F = 总节点力向量（外加载荷+等效节点力）。

### Q4 🟠 FEM是精确解还是近似解?

**EN:** FEM provides an **approximate solution**. The exact solution satisfies the governing PDE at EVERY material point (strong form), while FEM satisfies it only in a weighted-integral sense (weak form). The FEM displacement field is constrained by the assumed shape functions. In displacement-based FEM, the model is **stiffer** than the real structure, so displacements are **underestimated** (lower bound).

**中文:** FEM是近似解。精确解需在每个物质点满足PDE（强形式），FEM仅在加权积分意义上满足（弱形式）。位移型FEM偏硬（低估位移），解从下方收敛于精确解。

### Q5 🟢 节点(node)和单元(element)的定义

**EN:** A **node** is a discrete point in the mesh where displacements (DOFs) are defined and computed. An **element** is a subdomain of the structure bounded by nodes, within which the displacement field is interpolated from nodal values using shape functions.

**中文：** 节点是网格中定义和求解位移的离散点；单元是由节点围成的子区域，内部位移由节点位移通过形函数插值得到。

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### Q6 🟢 自由度(DOF)的定义

**EN:** A Degree of Freedom (DOF) is an independent displacement (or rotation) component at a node. In 2D plane problems each node has 2 DOFs ($u, v$); in 3D solids each node has 3 DOFs ($u, v, w$); for beams each node also has rotational DOFs ($\theta$).

**中文：** 自由度是节点处独立的位移（或转角）分量。平面问题每节点2个DOF($u, v$)；三维实体每节点3个DOF($u, v, w$)；梁单元还有转角DOF($\theta$)。

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# L2 — Preknowledge（预备知识）

## Q7 🟠 线弹性问题的三类基本方程

**EN:** The three sets of governing equations for linear elasticity are:

| Type | Equation | Meaning |
|---|---|---|
| Kinematic (Geometric) | $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$ | Displacement → Strain (6 equations) |
| Constitutive | $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ | Strain → Stress (6 equations) |
| Equilibrium | $\nabla^T \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}$ | Force balance (3 equations) |

Total: 15 equations for 15 unknowns (6 stresses + 6 strains + 3 displacements).

中文： 三类方程——(1)几何方程 $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$：位移求导得应变；(2)本构方程 $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$：应变映射为应力（Hooke定律）；(3)平衡方程 $\nabla^T \boldsymbol{\sigma} + \mathbf{f} = 0$：力平衡。共15个方程，15个未知量。

## Q8 🟠 最小势能原理

**EN:** The Principle of Minimum Potential Energy states that among all kinematically admissible displacement fields, the one that satisfies equilibrium minimizes the total potential energy:

$$\Pi_p = U + W = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV - \int_V \mathbf{u}^T \mathbf{f}\, dV - \int_S \mathbf{u}^T \mathbf{T}\, dS$$

The equilibrium condition is $\delta \Pi_p = 0$. Substituting $\mathbf{u} = \mathbf{N}\mathbf{d}$ and $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}$ yields the FEM equation $\mathbf{K}\mathbf{d} = \mathbf{F}$.

中文： 在所有运动学容许的位移场中，使总势能（应变能+外力势能）取最小值的那个满足平衡条件。$\delta \Pi_p = 0 \to$ 代入 $\mathbf{u} = \mathbf{N}\mathbf{d}, \boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d} \to$ 得FEM方程 $\mathbf{K}\mathbf{d} = \mathbf{F}$。

## Q9 ⭐ 单元刚度矩阵的一般公式

**EN:** The element stiffness matrix in its most general form is:

$$\boxed{\mathbf{k}_e = \int_{V_e} \mathbf{B}^T \mathbf{D} \mathbf{B}\, dV}$$

where $\mathbf{B}$ is the strain-displacement matrix ($\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}$), $\mathbf{D}$ is the constitutive (material) matrix ($\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$), and the integration is over the element volume.

中文： $\mathbf{k}_e = \int \mathbf{B}^T \mathbf{D} \mathbf{B}\, dV$ 是单元刚度矩阵的通式。B是应变-位移矩阵，D是材料本构矩阵。这是FEM最重要的公式之一，所有单元刚度矩阵都由它推导。

## Q10 ⭐ 位移插值关系

**EN:** In FEM, the displacement field within an element is interpolated from nodal displacements using shape functions:

$$\mathbf{u} = \mathbf{N}\mathbf{d}$$

The strain is then: $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}$, where $\mathbf{B}$ is the derivative of $\mathbf{N}$.

The stress is: $\boldsymbol{\sigma} = \mathbf{D}\mathbf{B}\mathbf{d}$.

中文： 三大基本关系——位移插值 $\mathbf{u} = \mathbf{N}\mathbf{d}$（N是形函数矩阵）；应变 $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}$（B是N的导数）；应力 $\boldsymbol{\sigma} = \mathbf{D}\mathbf{B}\mathbf{d}$。这是FEM推导的三步曲。

### Q11 ⭐ 刚度矩阵奇异性

**EN:** Before applying sufficient boundary conditions, the global stiffness matrix $\mathbf{K}$ is **singular** (determinant = 0, cannot be inverted). This is because the structure can undergo **rigid-body motion** — translating or rotating without developing any strain or stress. Sufficient constraints must be applied to eliminate all rigid-body modes before solving for displacements.

**中文：** 未施加足够约束前，K是奇异的（不可逆），因为结构可发生刚体位移（平动/转动不产生应变）。必须施加足够的位移约束消除所有刚体模态后，才能求解位移。

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### Q12 🔴 位移型FEM给出精确解的"下界"

**EN:** In displacement-based FEM, the model is **stiffer** than the actual structure because the assumed displacement field (shape functions) constrains the deformation. As a result, FEM displacements are **smaller in magnitude** than the exact solution — the FEM solution provides a **lower bound** and converges to the exact solution **from below** as the mesh is refined. This is known as the "stiffening effect" of displacement-based FEA. Note: this applies only to displacement-based formulations, not mixed or hybrid methods.

**中文：** 位移型FEM偏硬（stiffening effect），因为假定的形函数约束了变形模式，使结构比实际更"刚"。因此FEM位移解小于精确解（下界解），随网格加密从下方收敛。注意这只适用于位移型FEM，混合法和杂交法不适用。

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### Q13 🟢 对称矩阵与正交矩阵

**EN:** A symmetric matrix satisfies $\mathbf{A}^T = \mathbf{A}$ (e.g., $\mathbf{K}$, $\mathbf{D}$, $\mathbf{M}$ in FEM). An orthogonal matrix satisfies $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, meaning $\mathbf{A}^{-1} = \mathbf{A}^T$ (e.g., rotation/coordinate transformation matrices).

**中文：** 对称矩阵：$\mathbf{A}^T = \mathbf{A}$（如K,D,M都是对称的）；正交矩阵：$\mathbf{A}^T\mathbf{A} = \mathbf{I}$（如坐标变换矩阵）。利用对称性可节省一半以上的存储和计算量。

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## L3 — Direct Stiffness Method（直接刚度法）

### Q14 🟡 弹簧单元刚度矩阵

**EN:** For a spring element with stiffness $k$ connecting nodes $i$ and $j$:

$$\mathbf{k}_e = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}, \quad \begin{Bmatrix} f_i \\ f_j \end{Bmatrix} = \mathbf{k}_e \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

Note: a single unconstrained spring has a singular stiffness matrix (rigid-body mode).

**中文：** 弹簧单元刚度矩阵为 $k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$。单个未约束弹簧的刚度矩阵不可逆（刚体位移）。

### Q15 🟡 直接刚度法的装配思想

**EN:** The Direct Stiffness Method assembles the global stiffness matrix by:

1. Writing each element's local stiffness matrix in terms of its nodal DOFs
2. Mapping local DOF numbers to global DOF numbers
3. **Adding** (superposing) stiffness contributions at shared DOF positions

For example, two springs in series: element 1 (DOFs 1,2) and element 2 (DOFs 2,3). The global $\mathbf{K}$ is $3 \times 3$ with $K_{22} = k_1 + k_2$ (both elements contribute to the shared node).

**中文：** 直接刚度法按"对号入座"原则装配：将每个单元刚度矩阵按节点自由度编号叠加到全局刚度矩阵中。共享节点的刚度贡献相加。这是FEM装配的核心操作。

### Q16 🔴 总刚度矩阵的性质（4条）

**EN:** Properties of the global stiffness matrix $\mathbf{K}$:

1. **Symmetry**: $\mathbf{K}^T = \mathbf{K}$ (from Betti's reciprocal theorem / symmetric material matrix)
2. **Singularity** (before BCs): $\det(\mathbf{K}) = 0$ due to rigid-body modes
3. **Sparsity**: Most entries are zero — each node only connects to neighboring nodes
4. **Banded structure**: Non-zero entries cluster near the diagonal; bandwidth depends on node numbering

**中文：** 四大性质——(1)对称性（由本构对称性和Betti互等定理保证）；(2)奇异性（约束前，存在刚体位移）；(3)稀疏性（节点只与相邻节点连接，大量零元素）；(4)带状结构（非零元素集中在主对角线附近，带宽取决于节点编号）。

### Q17 🔴 减小刚度矩阵带宽的方法

**EN:** The bandwidth of $\mathbf{K}$ can be minimized by **optimizing the node numbering scheme**. Nodes should be numbered such that the maximum difference between connected node numbers is as small as possible. For elongated structures, number nodes across the shorter dimension first. This reduces computational cost (solution time $\propto N \cdot B^2$, where $B$ is the bandwidth).

**中文：** 优化节点编号顺序可减小带宽。应使相连节点的编号差尽可能小。对于细长结构，沿短方向优先编号。减小带宽可大幅降低求解时间（求解时间正比于 $N \times B^2$）。

### Q18 🟡 施加位移边界条件的方法

**EN:** Two common methods for applying displacement BCs:

1. **Partitioning method** (划行划列法): Partition $\mathbf{K}$ into free ($f$) and constrained ($c$) DOFs. Solve $\mathbf{K}_{ff}\mathbf{U}_f = \mathbf{F}_f - \mathbf{K}_{fc}\mathbf{U}_c$. Then reactions: $\mathbf{F}_c = \mathbf{K}_{cf}\mathbf{U}_f + \mathbf{K}_{cc}\mathbf{U}_c$.

2. **Penalty method** (置大数法): Add a very large number $C$ to the diagonal term $K_{ii}$ and replace $F_i$ with $C \cdot \bar{u}_i$ for the constrained DOF. This approximately enforces $u_i \approx \bar{u}_i$.

中文： 两种方法——(1)划行划列法：将K按自由度和约束自由度分块，求解约化方程组，再回代求反力；(2)置大数法（罚函数法）：在约束自由度对应的对角线元素上加一个大数C，并修改右端项，近似强制位移条件。

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### Q19 🔴 施加约束后K为何正定?

**EN:** After applying sufficient boundary conditions to eliminate all rigid-body modes, the reduced stiffness matrix $\mathbf{K}_{ff}$ becomes **positive definite** ($\mathbf{x}^T\mathbf{K}_{ff}\mathbf{x} > 0$ for any non-zero $\mathbf{x}$). This means: (1) the matrix is invertible; (2) the strain energy $U = \frac{1}{2}\mathbf{U}^T\mathbf{K}\mathbf{U} > 0$ for any non-rigid-body deformation; (3) Cholesky decomposition and other efficient solvers can be used.

中文： 消除刚体模态后，$\mathbf{K}_{ff}$ 正定——任何非零位移都产生正应变能（$U > 0$），矩阵可逆，可用Cholesky分解等方法高效求解。正定性也是解的唯一性的保证。

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### Q20 🟠 支座反力的计算

**EN:** After solving for unknown displacements $\mathbf{U}_f$, the reaction forces at constrained DOFs are:

$$\mathbf{F}_c = \mathbf{K}_{cf}\mathbf{U}_f + \mathbf{K}_{cc}\mathbf{U}_c$$

For zero prescribed displacement ($\mathbf{U}_c = \mathbf{0}$): $\mathbf{F}_c = \mathbf{K}_{cf}\mathbf{U}_f$.

中文： 反力 = 约束自由度对应的刚度行 × 所有位移。若约束位移为零，则 $\mathbf{F}_c = \mathbf{K}_{cf}\mathbf{U}_f$（已知位移为零时简化）。

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## L4 — Truss Problems（杆/桁架单元）

### Q21 ⭐ 杆单元刚度矩阵

**EN:** For a 1D bar element of length $L$, cross-sectional area $A$, and Young's modulus $E$:

$$\boxed{\mathbf{k}_e = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

中文：一维杆单元刚度矩阵为 $\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$。EA/L 是杆的轴向刚度。注意杆单元只能承受轴向拉压。

### Q22 🔴 杆单元不能承受什么力?

**EN:** A bar/truss element can ONLY carry **axial forces** (tension and compression). It CANNOT resist **shear forces** or **bending moments**. Each node has only translational DOFs along the bar axis. If the structure is subjected to transverse loads, a truss model alone is insufficient — beam or frame elements must be used.

中文：杆单元只能承受轴向拉压，不能承受剪力和弯矩。每个节点只有轴向平动自由度。若结构承受横向载荷，必须使用梁单元或框架单元。

### Q23 🟠 杆单元形函数和B矩阵

**EN:** For a 1D bar element, the linear shape functions are:

$$N_i(x) = 1 - \frac{x}{L}, \quad N_j(x) = \frac{x}{L}$$

The strain-displacement matrix $\mathbf{B}$ (1×2) is:

$$\mathbf{B} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

Since $\mathbf{B}$ is constant, strain $\varepsilon = \mathbf{Bd} = \frac{u_j - u_i}{L}$ is constant within the element.

中文：形函数 $N_i = 1 - x/L, N_j = x/L$（线性插值）；B矩阵 $[-1/L, 1/L]$ 为常数 → 单元内应变为常数。杆单元的应变 = 两端位移差/长度。

### Q24 🟠 二维桁架坐标变换

**EN:** For a 2D truss element oriented at angle $\theta$ from the global X-axis, define direction cosines $c = \cos\theta$, $s = \sin\theta$. The element stiffness in global coordinates (4×4) is:

$$\boxed{\mathbf{k}_e = \frac{EA}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}}$$

where $c = (x_j - x_i)/L$, $s = (y_j - y_i)/L$.

中文：通过方向余弦 $c = \cos\theta$, $s = \sin\theta$ 将局部坐标刚度变换到全局坐标。全局刚度矩阵为4×4（每节点2个DOF），含 $c^2, cs, s^2$ 项。变换原理：$\mathbf{k}_{global} = \mathbf{T}^T\mathbf{k}_{local}\mathbf{T}$。

### Q25 🔴 分布载荷的等效节点力

**EN:** For a bar element with distributed axial load $q(x)$, the equivalent nodal forces are calculated using the shape functions:

$$\mathbf{f}_s^e = \int_0^L \mathbf{N}^T q(x)\,dx$$

For **uniform load** $q$: $\mathbf{f}_s^e = \begin{Bmatrix} qL/2 \\ qL/2 \end{Bmatrix}$ (equal split).

For **linearly varying load** $q = Cx$: $\mathbf{f}_s^e = \begin{Bmatrix} CL^2/6 \\ CL^2/3 \end{Bmatrix}$ (1/3 to left, 2/3 to right).

The total equivalent force equals the total applied load ($\int_0^L q(x)\,dx$).

中文：等效节点力通过形函数积分得到：$\mathbf{f}_s^e = \int \mathbf{N}^T q\,dx$。均布：各得一半；线性分布：1/3给近端，2/3给远端。等效节点力的总和等于实际总载荷（静力等效）。

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## Q26 ⭐ 收敛性要求

**EN:** For convergence of FEA, two requirements must be satisfied:

1. **Completeness**: The shape functions must be able to represent **rigid-body displacements** (constant terms in the polynomial) and **constant strain states** (linear terms). This is verified by the **patch test**.
2. **Compatibility**: The displacement field must be **continuous across element boundaries**. For bar elements, this means $C^0$ continuity (displacement continuous, but its derivative — strain — may be discontinuous).

Together: **Completeness** + **Compatibility** → **Convergence** (as mesh is refined).

中文：收敛的两个必要条件——(1)完备性：形函数能表示刚体位移（常数项）和常应变（线性项），通过分片试验验证；(2)相容性：位移在单元边界连续。杆单元满足$C^0$连续（位移连续，导数/应变可不连续）。完备+相容+稳定→收敛。

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## Q27 🔴 $C^0$ vs $C^1$ 连续性

**EN:** Continuity classification for FEM elements:

- **$C^0$ continuity**: The field variable itself is continuous across element boundaries, but its first derivative may be discontinuous. Examples: bar elements, plane stress/strain elements, 3D solid elements. (Displacement is continuous, strain/stress may jump.)
- **$C^1$ continuity**: Both the field variable AND its first derivative are continuous across boundaries. Example: beam elements (both deflection $v$ AND slope $\theta = dv/dx$ are continuous).

中文：$C^0$连续——场变量本身连续，但一阶导数可间断（如杆单元：位移连续，应变不连续）；$C^1$连续——场变量及其一阶导数都连续（如梁单元：挠度和转角都连续）。板的Kirchhoff理论要求$C^1$连续。

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## Q28 🟡 梁挠度插值多项式的阶次（易错）

**EN:** The beam element uses a **cubic (3rd-order) polynomial** for deflection:

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

This is exact when no distributed load acts on the element ($EI\,d^4v/dx^4 = 0$). However, under a **uniform distributed load** $q$, the exact solution is **4th-order** ($v \propto x^4$ term from integrating $q/EI$ four times). The cubic FEM approximation is therefore NOT exact for distributed loads — the solution matches exactly only at the nodes, while interior points have some error.

中文：⚠️ 易错点——梁单元位移插值是**三次**多项式。无分布载荷时三次是精确解；有均布载荷时，精确解是**四次**（因为 $EI\,v'''' = q$，积分四次产生 $x^4$ 项），FEM的三次解仅是近似，只在节点处匹配。考试常设陷阱。

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## Q29 ⭐ 梁单元等效节点力

**EN:** For a beam element under uniform transverse load $q$:

$$\mathbf{f}_e = \begin{Bmatrix} qL/2 \\ qL^2/12 \\ qL/2 \\ -qL^2/12 \end{Bmatrix} \begin{matrix} \leftarrow v_i \\ \leftarrow \theta_i \\ \leftarrow v_j \\ \leftarrow \theta_j \end{matrix}$$

These are the **fixed-end forces**: shear $qL/2$ at each end, and moments $qL^2/12$ (counter-clockwise at $i$, clockwise at $j$). Sign conventions depend on the local coordinate system.

中文：均布载荷q下，梁单元等效节点力 = $\{qL/2, qL^2/12, qL/2, -qL^2/12\}$。物理含义：两端各承担一半剪力，固端弯矩 $qL^2/12$。注意符号取决于坐标方向约定。

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## Q30 🔴 梁单元等效节点力与支座反力的区别

**EN:** A critical distinction in beam problems: when computing **reaction forces** at supports after solving for displacements, the effect of equivalent nodal loads must be properly accounted for. The total reaction = stiffness contribution + equivalent load contribution.

Specifically, for a beam with distributed load $q$, the element equation is $\mathbf{k}_e\mathbf{d}_e = \mathbf{f}_{ext} + \mathbf{f}_{eq}$, where $\mathbf{f}_{eq}$ is the fixed-end force vector. When computing reactions at constrained nodes, the fixed-end force contribution at that node must be included.

中文：⚠️ 常错——计算支座反力时，不仅要考虑刚度×位移的贡献，还要加上等效节点力在支座处的分量。因为分布载荷被转化为等效节点力后，这些"等效节点力"也参与了力的平衡。如均布载荷下的固支梁，即使位移为零，固端反力也不为零。

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## Q31 ⭐ 梁弯曲正应力公式

**EN:** The bending normal stress in a beam is:

$$\sigma = -\frac{My}{I}$$

where $M$ is the bending moment, $y$ is the distance from the neutral axis, and $I$ is the second moment of area. The negative sign indicates compressive stress on the positive-$y$ side when $M$ is positive (conventional beam sign convention).

中文：$\sigma = -My/I$，M是弯矩，y是距中性轴的距离，I是截面惯性矩。负号表示正弯矩时上压下拉（或取决于坐标约定）。

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## Q32 🔴 平面框架单元

**EN:** A 2D frame element combines axial and bending behavior. Each node has **3 DOFs**: $\{u_i, v_i, \theta_i\}$. The element has 6 DOFs total. The 6×6 stiffness matrix is assembled by combining:

- Axial stiffness ($EA/L$ terms at DOF positions 1 and 4)
- Bending stiffness ($12EI/L^3$, $6EI/L^2$, $4EI/L$, etc. at DOF positions 2,3,5,6)

The full matrix is given in L5 course notes. For 3D frames, each node has 6 DOFs (3 translations + 3 rotations), requiring a 12×12 stiffness matrix including axial, torsional ($GJ/L$), and bending in two planes.

中文：平面框架单元 = 杆单元（轴向）+ 梁单元（弯曲）的组合，每节点3个DOF（$u, v, \theta$），共6个。三维框架单元每节点6个DOF（3个平动+3个转动），共12×12刚度矩阵（含轴向+扭转$GJ/L$+两个方向的弯曲）。

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## Q33 🟠 梁铰接点处理

**EN:** When a hinge (pin connection) exists within a beam, the bending moment at the hinge is zero and the slope is discontinuous. To model this:

1. **Static condensation**: Eliminate the rotational DOF at the hinge from the stiffness matrix. For a hinge at the right end, the condensed stiffness matrix reduces from 4×4 to 3×3.
2. **Hinge element**: Use a specially derived element with the zero-moment condition pre-applied.

The condensed stiffness for a beam with hinge at right end (DOFs: $v_i, v_j, \theta_j$):

$$\mathbf{k} = \frac{3EI}{L^3}\begin{bmatrix} 1 & L & -1 \\ L & L^2 & -L \\ -1 & -L & 1 \end{bmatrix}$$

中文： 铰接点弯矩为零，转角不连续。处理方法：用静力凝聚消去铰接处的转动自由度。右端铰接的梁单元刚度矩阵缩聚为3×3。

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## L7 — 2D Problems（二维问题）

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### Q34 ⭐ 平面应力 vs 平面应变

**EN: Plane stress** applies to **thin** structures (plates with in-plane loading):

- Condition: $\sigma_z = \tau_{xz} = \tau_{yz} = 0$
- BUT $\varepsilon_z \neq 0$ (the plate is free to contract/expand in thickness)
- Typical application: thin plates with in-plane forces, thin-walled structures

**Plane strain** applies to **thick/long** structures (cross-section of long bodies):

- Condition: $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$
- BUT $\sigma_z \neq 0$ (constrained in the longitudinal direction)
- Typical application: long dams, tunnels, retaining walls, shafts

中文： ⚠️ 高频考点——平面应力（薄板）： $\sigma_z = 0$ 但 $\varepsilon_z \neq 0$，板厚方向自由变形，用于薄板面内载荷；平面应变（长厚体）： $\varepsilon_z = 0$ 但 $\sigma_z \neq 0$，纵向受约束，用于长坝、隧道、挡土墙等。两个的D矩阵不同！

---

### Q35 🟠 平面应力D矩阵

**EN:** Plane stress constitutive matrix (3×3):

$$\boxed{\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}$$

Stress and strain vectors: $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \tau_{xy}\}^T, \boldsymbol{\varepsilon} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}^T$.

中文： 平面应力D矩阵（3×3），因子为 $E/(1-\nu^2)$。对应应力/应变分量各3个（$\sigma_z = 0$所以不出现）。

---

### Q36 🟠 平面应变D矩阵

**EN:** Plane strain constitutive matrix (3×3):

$$\boxed{\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}}$$

Alternatively: use the plane stress D-matrix but replace $E$ with $E' = E/(1-\nu^2)$ and $\nu$ with $\nu' = \nu/(1-\nu)$.

中文： 平面应变D矩阵（3×3），因子为 $E/[(1+\nu)(1-2\nu)]$。对角项为 $1-\nu$ 而非1。等价变换： $E \to E/(1-\nu^2), \nu \to \nu/(1-\nu)$。

---

### Q37 🟠 二维应变-位移关系

**EN:** In 2D (plane problems), the three strain components are:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

In matrix form:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

**中文：** 二维应变——正应变 $\varepsilon_x = \partial u/\partial x, \varepsilon_y = \partial v/\partial y$；剪应变 $\gamma_{xy} = \partial u/\partial y + \partial v/\partial x$（工程剪应变，非张量分量）。

---

## Q38 ⭐ CST常应变三角形

**EN:** The **Constant Strain Triangle (CST, T3)** is a 3-node triangular element with linear displacement functions:

$$u = a_1 + a_2 x + a_3 y, \quad v = a_4 + a_5 x + a_6 y$$

Since displacements are linear, their derivatives (strains) are **constant** throughout the element. The B-matrix is constant:

$$\mathbf{k}_e = tA\,\mathbf{B}^T\mathbf{D}\mathbf{B}$$

**Advantages:** Simple derivation; good for complex geometries.

**Disadvantages:** Low accuracy (constant stress); overly stiff in bending; requires dense mesh for good results.

**中文：** CST（常应变三角形，T3）：3节点，位移线性 → 应变为常数（故名"常应变"）。优点：简单、几何适应性强；缺点：精度低、弯曲问题中偏硬、需密网格。刚度矩阵 $\mathbf{k}_e = tA\mathbf{B}^T\mathbf{D}\mathbf{B}$，无需数值积分。

---

## Q39 ⭐ CST形函数的一阶导数

**EN:** For a CST element, shape functions are:

$$N_i = \frac{\alpha_i + \beta_i x + \gamma_i y}{2A}$$

where $A$ is the element area, and $\beta_i = y_j - y_m, \gamma_i = -(x_j - x_m)$ (cyclic permutation).

The first derivatives $\partial N_i/\partial x = \beta_i/(2A)$ and $\partial N_i/\partial y = \gamma_i/(2A)$ are **constants**. Therefore, the B-matrix is constant.

For a **quadrilateral element**, the derivatives $\partial N_i/\partial x$ and $\partial N_i/\partial y$ involve $\mathbf{J}^{-1}$ (Jacobian inverse), which generally varies with position → derivatives are **not constant**.

**中文：** CST的形函数导数是常数（因位移线性）；四边形单元的形函数导数通过Jacobian逆矩阵变换得到，随位置变化，不是常数。这就是"三角形常应变，四边形非常应变"的原因。

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## Q40 🔴 CST的优缺点总结

**EN: CST advantages:** (1) Simple formulation — easy to program; (2) Good for complex geometries — triangles can mesh any shape; (3) No numerical integration needed; (4) Good for preliminary/quick analysis.

**CST disadvantages:** (1) Constant stress/strain — poor accuracy in regions with high gradients; (2) Overly stiff in bending — requires very fine mesh for beam/plate problems; (3) Lower accuracy per DOF compared to quadrilateral or higher-order elements; (4) Not suitable for stress concentration regions (holes, corners).

**Recommendation:** Use CST in areas with small strain gradients, mesh transition zones, or quick preliminary analysis. Avoid CST at stress concentrations — use higher-order elements or very fine mesh.

**中文：** 优点：简单、几何适应性好、无需数值积分。缺点：常应力精度低、弯曲问题偏硬、应力集中处表现差。建议：应变梯度小处用CST，应力集中处用高阶单元或加密网格。

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## Q41 🔴 不同二维单元的比较

**EN:** Comparison of 2D element types:

| Element | Nodes | Displacement | Strain | Stress |
|---|---|---|---|---|
| T3 (CST) | 3 | Linear | Constant | Constant |
| Q4 (Bilinear Quad) | 4 | Bilinear | Linear (varies) | Linear (varies) |
| T6 (LST) | 6 | Quadratic | Linear (varies) | Linear (varies) |
| Q8 (Quadratic Quad) | 8 | Quadratic/cubic | Quadratic | Quadratic |

**Rule of thumb:** For the same number of DOFs, quadrilaterals generally perform better than triangles. Higher-order elements give better accuracy per DOF but have higher bandwidth.

中文： T3（常应变）、Q4（双线性，应变线性变化）、T6（位移二次，应变线性变化）、Q8（位移三次，应变二次变化）。经验法则：同样DOF下，四边形优于三角形；高阶单元精度更高但带宽更大。

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### Q42 🔴 平面应力和平面应变的适用范围

**EN: Plane stress** is appropriate for:

- Thin plates with in-plane loading (thickness $\ll$ in-plane dimensions)
- Thin-walled pressure vessels (membrane stresses)
- Sheet metal forming analysis
- Web of an I-beam (local analysis)

**Plane strain** is appropriate for:

- Very long/thick structures with uniform cross-section
- Dams, tunnels, retaining walls
- Cylindrical rollers under compression
- Soil/geotechnical cross-sections

中文： 平面应力——薄板面内加载、薄壁容器、板料成形；平面应变——长坝、隧道、挡土墙、轧辊、岩土截面。判断标准：厚度方向的变形是否被约束。

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## L8 — Axisymmetric Elements（轴对称单元）

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### Q43 🔴 轴对称的四个必要条件

**EN:** For a problem to be treated as axisymmetric, ALL four conditions must be satisfied:

1. **Geometry** — the structure is a solid of revolution (rotationally symmetric about an axis)
2. **Material properties** — axisymmetric (same in all circumferential directions)
3. **Boundary conditions** — axisymmetric (no variation in $\theta$)
4. **Loading** — axisymmetric (no variation in $\theta$)

If any condition is violated, a full 3D or other simplified model must be used.

中文： 轴对称必须同时满足几何、材料、边界条件、载荷四个方面的轴对称性。任何一个不满足都不能简化为轴对称模型。典型结构：轴、管道、压力容器、回转体。

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### Q44 🟠 轴对称坐标系统

**EN:** Axisymmetric problems use cylindrical coordinates $(r, \theta, z)$. Due to axisymmetry:

- $\frac{\partial(\cdot)}{\partial\theta} = 0$ — nothing varies with circumferential angle
- $v_\theta = 0$ — no circumferential displacement
- Displacements are functions of $r$ and $z$ only: $u = u(r, z), w = w(r, z)$

**中文：** 柱坐标 $(r,\theta,z)$。轴对称条件：$\partial(\cdot)/\partial\theta=0$（无不随θ变化），$v_\theta=0$（无环向位移）。位移仅为r和z的函数，问题从3D降为2D（r-z平面）。

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### Q45 ⭐ 环向应变 $\varepsilon_\theta = u/r$

**EN:** The distinguishing feature of axisymmetric problems is the **hoop (circumferential) strain**:

$$\boxed{\varepsilon_\theta = \frac{u}{r}}$$

where $u$ is the radial displacement and $r$ is the radial coordinate. This strain arises because radial expansion/contraction changes the circumference. The axisymmetric strain vector has **4 components** (one more than plane problems):

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$

**中文：** 轴对称的标志性特征——环向应变 $\varepsilon_\theta = u/r$。当结构径向变形时，周长改变产生环向应变。轴对称有4个应变分量（比平面问题多了 $\varepsilon_\theta$）。

---

### Q46 ⭐ 3节点轴对称单元不是常应变单元

**EN:** Although the 3-node axisymmetric element uses the same shape functions as CST in the $r$-$z$ plane, it is **NOT a constant strain element**. The hoop strain $\varepsilon_\theta = u/r$ contains $1/r$ terms (through the shape function $N_i/r = (\alpha_i + \beta_i r + \gamma_i z)/(2Ar)$), which vary with radial position. Therefore, the B-matrix is not constant, and the strains vary within the element. This is a common exam trap.

**中文：** ⚠️ 高频易错——3节点轴对称单元虽然在r-z平面看像CST三角形，但**不是**常应变单元！因为 $\varepsilon_\theta = u/r$ 含有 $1/r$ 项，导致B矩阵随r变化，应变在单元内不是常数。这是与平面CST的本质区别。

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### Q47 🟠 轴对称D矩阵（4×4）

**EN:** The axisymmetric constitutive matrix (4×4):

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Corresponding stress and strain vectors: $\{\sigma_r, \sigma_z, \sigma_\theta, \tau_{rz}\}^T$ and $\{\varepsilon_r, \varepsilon_z, \varepsilon_\theta, \gamma_{rz}\}^T$.

**中文：** 轴对称D矩阵为4×4，含 $(1-\nu)/((1+\nu)(1-2\nu))$ 因子（与3D问题类似）。

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### Q48 🔴 轴对称刚度积分中的 $2\pi r$

**EN:** The volume element in axisymmetric coordinates is $dV = r\,dr\,d\theta\,dz$. Integrating over $\theta$ from 0 to $2\pi$ gives:

$$dV = 2\pi r\,dA \quad (\text{where } dA = dr\,dz \text{ is the area in the r-z plane})$$

Therefore, the element stiffness integral for axisymmetric elements is:

$$\boxed{\mathbf{k}_e = \int_{A_e} 2\pi r\,\mathbf{B}^T\mathbf{D}\mathbf{B}\,dA}$$

This is fundamentally different from plane problems ($t\,dA$). The $r$ inside the integral makes analytical integration difficult, so numerical integration is generally required.

**中文：** ⚠️ 轴对称的刚度积分中有 $2\pi r$（而不是平面问题的 $t$）！体积微元 $dV = 2\pi r\,dA$。这个额外的 $r$ 因子使得即使B矩阵近似常数，积分也不是简单的面积乘积，通常需要数值积分。

### Q49 🔴 轴对称与平面CST的全面对比

**EN:**

| Feature | Plane CST | 3-Node Axisymmetric |
|---|---|---|
| Coordinate system | $x$-$y$ (Cartesian) | $r$-$z$ (Cylindrical) |
| Strain components | 3 ($\varepsilon_x, \varepsilon_y, \gamma_{xy}$) | 4 (adds $\varepsilon_\theta$) |
| Hoop strain | None | $\varepsilon_\theta = u/r$ |
| D-matrix | 3×3 | 4×4 |
| Strain variation | **Constant** | **NOT constant** ($1/r$ terms) |
| Volume element | $t\,dA$ | $2\pi r\,dA$ |
| Numerical integration | Not needed for CST | Generally required |

**中文：** 这个对比表是必背内容。最关键的区别：轴对称有环向应变，应变不是常数，积分含 $2\pi r$。

## L8 Practical — 建模实践

### Q50 🔴 长宽比(Aspect Ratio)

**EN:** The **Aspect Ratio** is defined as:

$$\text{Aspect Ratio} = \frac{\text{longest dimension}}{\text{shortest dimension}}$$

As the aspect ratio increases, solution accuracy **decreases**. The effect is most severe in regions of **high stress gradient**. In regions where the stress gradient is nearly zero, a larger aspect ratio may be tolerable. The ideal aspect ratio is close to 1.

**中文：** 长宽比 = 最长尺寸/最短尺寸。长宽比越大，精度越低。应力梯度越大的区域对长宽比越敏感。理想长宽比接近1。应避免过大长宽比、过度扭曲、角度过大或过小的单元。

### Q51 ⭐ 需要在哪些位置布置节点?

**EN:** Nodes (and mesh lines) must be placed at:

1. **Geometric discontinuities** — abrupt changes in geometry, thickness changes, corners
2. **Loading points** — concentrated forces, distributed load start/end
3. **Material interfaces** — changes in material properties
4. **Boundary condition changes** — transitions between different support types
5. **Stress concentration regions** — holes, notches, sharp corners, fillets

**中文：** 节点必须布置在：几何突变处、载荷作用点、材料界面、边界条件变化处、应力集中区（孔、槽、尖角）。在变化剧烈处应加密网格。

### Q52 🔴 不同单元连接时的自由度匹配

**EN:** When connecting different element types (e.g., plane elements with beam elements), the DOF compatibility must be carefully considered. Plane elements have only translational DOFs ($u, v$), while beam elements also have rotational DOFs ($\theta$). A direct connection means the plane element cannot transmit the moment associated with the beam's rotational DOF. Solutions include:

- Constraint equations linking DOFs
- Transition elements
- Avoiding direct connections (use consistent element types throughout)

中文：平面单元只有平动DOF($u, v$)，梁单元还有转动DOF($\theta$)。直接连接时，平面单元不能传递梁的弯矩。解决方案：约束方程、过渡单元、或使用一致的元素类型。这是建模中的常见错误来源。

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## Q53 ⭐ 对称性的使用条件和方法

**EN:** Symmetry can be exploited to reduce model size when ALL of the following are symmetric:

1. Geometry
2. Material properties
3. Boundary conditions
4. Loading

**Symmetry BCs:** On the symmetry plane, the **normal displacement component is zero**; tangential displacement is free.

**Reduction:** One symmetry line → half model; two orthogonal symmetry lines → quarter model.

⚠️ **CAUTION:** Do NOT use symmetry blindly for **vibration and buckling** analysis. Symmetric structures can have **anti-symmetric mode shapes** that would be completely missed by a symmetric model.

中文：必须四者（几何、材料、BC、载荷）都对称才能利用对称性。对称面BC：法向位移=0。一对称线减半，两正交对称线减至四分之一。⚠️ 振动/屈曲分析不能盲目用对称，会遗漏反对称模态！

---

## Q54 🔴 静力凝聚(Static Condensation)

**EN:** Static condensation is a technique to **eliminate internal DOFs** without losing accuracy, reducing the system size. Given partitioned equations:

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ab} \\ \mathbf{K}_{ba} & \mathbf{K}_{bb} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_a \\ \mathbf{U}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_a \\ \mathbf{F}_b \end{Bmatrix}$$

Where $a$ = external (retained) DOFs, $b$ = internal (condensed) DOFs:

$$\mathbf{U}_b = \mathbf{K}_{bb}^{-1}(\mathbf{F}_b - \mathbf{K}_{ba}\mathbf{U}_a)$$

Substituting back gives the condensed equation:

$$\boxed{(\mathbf{K}_{aa} - \mathbf{K}_{ab}\mathbf{K}_{bb}^{-1}\mathbf{K}_{ba})\mathbf{U}_a = \mathbf{F}_a - \mathbf{K}_{ab}\mathbf{K}_{bb}^{-1}\mathbf{F}_b}$$

Common use: condensing interior nodes of subdivided quadrilaterals, creating superelements.

中文：静力凝聚通过消去内部自由度来减小方程规模，但不损失精度。常用于消去四边形内部节点（4个三角形→5节点→凝聚为4节点）。凝聚后的刚度矩阵 $\mathbf{K}_{aa} - \mathbf{K}_{ab}\mathbf{K}_{bb}^{-1}\mathbf{K}_{ba}$ 称为等效刚度矩阵。

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## Q55 🟡 子结构(Substructure)思想

**EN:** A substructure (superelement) treats a portion of a structure as a single component. The interior DOFs are condensed to boundary DOFs via static condensation. Advantages: handles large problems, different teams can model different components, partial redesign requires only partial reanalysis, efficient for local nonlinearities. Disadvantages: increased file management, approximate for dynamic problems.

中文：子结构 = 将复杂结构的一部分视为一个"超级单元"，内部DOF凝聚到边界DOF。优点：适合大型结构、可分工建模、局部修改只需重新分析局部。缺点：文件管理复杂、动力问题有近似性。

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## Q56 ⭐ 有限元解的三大误差来源

**EN:** The three major errors in FEM:

1. **Discretization Error（离散误差）**: Error from replacing the continuous structure with a finite number of elements. Decreases as the mesh is refined.
2. **Modeling Error（建模误差）**: Error from idealizing the physical problem — geometric simplifications, material model assumptions (e.g., linear elastic vs. plastic), boundary condition idealizations.

3. **Numerical Error（数值误差）**: Error from finite-precision computation — round-off error, numerical integration error (insufficient Gauss points), error in solving linear systems.

中文：(1)离散误差——用有限元离散化替代连续体，网格越粗误差越大；(2)建模误差——物理问题理想化带来的误差（几何简化、材料假设、边界条件理想化）；(3)数值误差——计算精度有限导致的舍入误差、数值积分误差、方程求解误差。

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### Q57 ⭐ 网格收敛性(Mesh Convergence)

**EN:** Mesh convergence is a measure of the number of **elements** required in a model to ensure that the results of an analysis are not affected by changing the size of the **mesh**. The procedure: (1) create an initial mesh; (2) refine the mesh; (3) compare key results (displacements, stresses); (4) if results change significantly, refine further; (5) when further refinement produces negligible change → convergence achieved.

中文：网格收敛性指：逐步加密网格后，关键结果趋于稳定，不再随网格尺寸减小而明显变化。标准做法：加密→比较→再加密→直至结果稳定。位移收敛通常比应力收敛更快。

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### Q58 🔴 h-refinement vs p-refinement

**EN:** Three types of mesh refinement:

1. **h-refinement**: Reduce element size $h$ — increase the number of elements. Most common approach.
2. **p-refinement**: Increase the polynomial order $p$ of shape functions on the same mesh (e.g., CST → LST). Improves accuracy without changing the mesh.
3. **r-refinement**: Reposition nodes while keeping the same number of elements and DOFs — concentrate nodes in high-gradient regions.
4. **hp-refinement**: Combination of h- and p-refinement — produces the best convergence rate.

中文：h-细化（缩小单元尺寸）、p-细化（提高形函数阶次）、r-细化（重布节点位置）、hp-细化（组合法，收敛最快）。FEM收敛性分析中h-细化最常用。

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### Q59 🔴 FEM应力/应变的精度特点

**EN:** In displacement-based FEM:

1. **Displacement** has the **highest accuracy** and is continuous across element boundaries.
2. **Strain and stress** have **lower accuracy** (derived from displacement derivatives) and are generally **discontinuous** across element boundaries.
3. Stresses/strains are **most accurate at Gauss integration points** (superconvergent points).
4. Nodal stresses are obtained by **extrapolation** from Gauss points or by **averaging** (stress smoothing) from adjacent elements.
5. Simple averaging: $\boldsymbol{\sigma}_P = \frac{1}{n}\sum \boldsymbol{\sigma}_P^{(i)}$ or area-weighted averaging.

中文：位移精度 > 应力/应变精度（因为应力/应变来自位移的导数）。位移在单元间连续，应力/应变不连续。Gauss积分点处应力最准确（超收敛点）。节点应力通过外推或平均得到（应力平滑）。

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### Q60 🟠 直接法 vs 迭代法求解

**EN: Direct methods** (Gauss elimination): Solution time $\propto N \cdot B^2$ ($N$ = matrix dimension, $B$ = bandwidth). Good for small-to-medium problems, slender structures (small bandwidth), multiple load cases. **Iterative methods**: Reduced storage, good for large problems, bulky structures (faster convergence), but unknown solution time and need resolving for different load cases.

中文：直接法（Gauss消元）：求解时间正比于 $N \times B^2$，适合中小型、细长结构、多载荷工况；迭代法：存储量小，适合大型、块状结构，但求解时间不可预测，不同载荷需重新求解。

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## L9 — Isoparametric Formulation（等参单元）

### Q61 ⭐ "等参(isoparametric)"的含义

**EN:** "Isoparametric" means that the **same shape functions** are used to interpolate both the **geometry** (coordinates) AND the **displacement** field:

- Geometry: $x = \sum N_i(s,t)x_i, y = \sum N_i(s,t)y_i$
- Displacement: $u = \sum N_i(s,t)u_i, v = \sum N_i(s,t)v_i$

This is the defining characteristic: geometry and displacement use identical interpolation. If geometry uses fewer nodes than displacement → "subparametric"; if more → "superparametric."

**中文：** "等参" = 几何坐标插值和位移插值使用**同一组**形函数。几何：$x = \sum N_i x_i, y = \sum N_i y_i$；位移：$u = \sum N_i u_i, v = \sum N_i v_i$。这是等参单元的**定义性特征**。

### Q62 ⭐ 4节点四边形形函数

**EN:** The 4-node quadrilateral shape functions in natural coordinates $(s,t) \in [-1,1] \times [-1,1]$:

$$\boxed{N_i = \frac{1}{4}(1+s_i s)(1+t_i t)}$$

where $(s_i, t_i)$ are the natural coordinates of node $i$:

- $N_1$ at $(-1,-1)$: $N_1 = \frac{1}{4}(1-s)(1-t)$
- $N_2$ at $(+1,-1)$: $N_2 = \frac{1}{4}(1+s)(1-t)$
- $N_3$ at $(+1,+1)$: $N_3 = \frac{1}{4}(1+s)(1+t)$
- $N_4$ at $(-1,+1)$: $N_4 = \frac{1}{4}(1-s)(1+t)$

Properties: $N_i = 1$ at node $i$, $N_i = 0$ at all other nodes; $\sum N_i = 1$.

**中文：** 四节点四边形形函数为 $N_i = \frac{1}{4}(1+s_i s)(1+t_i t)$。自然坐标范围 $[-1,1] \times [-1,1]$。$N_i$在节点i为1、其他节点为0、和为1。

### Q63 ⭐ Jacobian矩阵

**EN:** The Jacobian matrix $\mathbf{J}$ maps derivatives from natural coordinates $(s,t)$ to global coordinates $(x,y)$:

$$\boxed{\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} \sum \frac{\partial N_i}{\partial s} x_i & \sum \frac{\partial N_i}{\partial s} y_i \\ \sum \frac{\partial N_i}{\partial t} x_i & \sum \frac{\partial N_i}{\partial t} y_i \end{bmatrix}}$$

The derivative transformation is:

$$\begin{Bmatrix} \partial N_i/\partial s \\ \partial N_i/\partial t \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \partial N_i/\partial x \\ \partial N_i/\partial y \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \partial N_i/\partial x \\ \partial N_i/\partial y \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \partial N_i/\partial s \\ \partial N_i/\partial t \end{Bmatrix}$$

Area transformation: $dx\,dy = |\mathbf{J}|\,ds\,dt$.

**中文：** Jacobian矩阵 $\mathbf{J}$ 将自然坐标导数映射为全局坐标导数。$\mathbf{J}$ 的每个元素 = $\sum \frac{\partial N_i}{\partial s} x_i$ 等。导数变换：全局导数 = $\mathbf{J}^{-1}$ × 自然导数。面积变换：$dxdy = |\mathbf{J}|dsdt$。

### Q64 ⭐ $|\mathbf{J}| > 0$ 的有效性条件

**EN:** For a valid isoparametric element, the Jacobian determinant must be **positive** everywhere:

$$|\mathbf{J}| > 0 \quad \text{at all points } (s,t) \in [-1,1]^2$$

$|\mathbf{J}| \leq 0$ indicates:

- The element is **excessively distorted** or **concave**

- One-to-one mapping between $(s, t)$ and $(x, y)$ is destroyed
- Interior angles exceed 180°
- Node ordering is wrong (not counterclockwise)

For convex quadrilaterals with proper node ordering, $|\mathbf{J}| > 0$ is guaranteed.

中文：⚠️ $|\mathbf{J}| > 0$ 是等参单元有效的充要条件。$|\mathbf{J}| \leq 0$ 意味着单元过度扭曲、凹四边形、节点顺序错误。对于凸四边形和正确的逆时针顺序，$|\mathbf{J}| > 0$ 自动满足。

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### Q65 🔴 $|\mathbf{J}| < 0$ 的物理含义

**EN:** When $|\mathbf{J}| < 0$, the natural-to-global coordinate mapping is no longer one-to-one. This means:

1. A single point in natural coordinates maps to multiple points in global coordinates (or vice versa)
2. The element has "folded over" onto itself
3. Interior angles exceed 180° (concave quadrilateral)
4. The element stiffness matrix may contain negative or erroneous entries
5. Solution results from such elements are **meaningless**

**Prevention:** Maintain convex element shapes, use proper node ordering (counterclockwise), avoid excessive distortion.

中文：$|\mathbf{J}| < 0$ 意味着映射不再是一对一的——单元"折叠"了，内角超过180°（凹四边形），刚度矩阵可能含负值或错误项，结果毫无意义。预防：保持凸形状、正确节点顺序（逆时针）、避免过度扭曲。

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### Q66 🔴 8节点四边形角节点形函数的构造方法

**EN:** For 8-node quadrilateral (Q8, serendipity element), the corner shape functions are constructed through a **correction procedure**:

1. **Mid-edge shape functions** (for nodes 5-8):
   - $N_5 = \frac{1}{2}(1-s^2)(1-t)$ (between nodes 1 and 2)
   - $N_6 = \frac{1}{2}(1+s)(1-t^2)$ (between nodes 2 and 3)
   - $N_7 = \frac{1}{2}(1-s^2)(1+t)$ (between nodes 3 and 4)
   - $N_8 = \frac{1}{2}(1-s)(1-t^2)$ (between nodes 4 and 1)
2. **Start with** the standard 4-node bilinear shape functions: $\hat{N}_i = \frac{1}{4}(1+s_i s)(1+t_i t)$
3. **Correct** each corner function by subtracting half of the adjacent mid-edge functions:

$$N_1 = \hat{N}_1 - \frac{1}{2}N_5 - \frac{1}{2}N_8$$

   and similarly for $N_2, N_3, N_4$.

This ensures corner nodes still satisfy $N_i = 1$ at node $i$ and $N_i = 0$ at all other nodes.

中文：Q8角节点形函数不能直接用双线性形函数，需要修正：角节点函数 = 4节点双线性形函数 − 1/2×(相邻两边中点形函数)。这保证了角节点处 $N_i = 1$、其他节点处 $N_i = 0$ 的性质。

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## L10 — Numerical Integration（数值积分）

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### Q67 ⭐ Gauss积分精度公式

**EN:** With $n$ Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly:

$$\boxed{\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} W_i f(x_i) \quad \text{exact for } f(x) \text{ of degree } \leq 2n-1}$$

Examples:

- $n = 1$ (1 point): exact for linear ($p = 1$)
- $n = 2$ (2 points): exact for cubic ($p = 3$)
- $n = 3$ (3 points): exact for quintic ($p = 5$)

中文： $n$ 点 Gauss 积分精确到 $2n - 1$ 次多项式。1点精确1次，2点精确3次，3点精确5次。这是Gauss积分比Newton-Cotes更高效的根本原因。

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### Q68 🔴 Newton-Cotes积分精度

**EN:** Newton-Cotes formulas use **equally spaced** sampling points. With $n$ points, they are exact for polynomials of degree $n - 1$ (or $n$ for even $n$ in some variants). This is much less efficient than Gauss quadrature for the same number of points. Comparison: With 3 points — Gauss exact to degree 5, Newton-Cotes exact to degree 2 (Simpson's rule) or 3. FEM programs generally prefer Gauss integration for efficiency.

中文： Newton-Cotes用等距积分点，$n$点精确 $n - 1$ 次（或n次），效率远低于Gauss。3点Gauss精确5次，3点Newton-Cotes仅精确2-3次。有限元优先使用Gauss积分。

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### Q69 🔴 Gauss vs Newton-Cotes 对比

**EN:**

| Feature | Gauss | Newton-Cotes |
|---|---|---|
| Sampling points | **Non-equally spaced** | **Equally spaced** |
| Boundary points sampled? | No | Yes |
| Accuracy ($n$ points) | $2n - 1$ degree | $n - 1$ (or $n$) degree |
| Efficiency | **Higher** | Lower |
| Usage in FEM | **Preferred** | Rare (plasticity starting from surfaces) |

Gauss is more efficient because it optimizes both point **locations** AND **weights**, while Newton-Cotes fixes point locations and optimizes only weights. However, in some applications like plasticity where yielding starts at element surfaces, Newton-Cotes may be used because it samples boundary points.

中文： Gauss用非等距点（内部点+优化权重），精度远高于等距的Newton-Cotes。有限元优先用Gauss。但Newton-Cotes采样边界点，在塑性（从表面开始屈服）中有应用。

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### Q70 ⭐ 4节点和8节点等参单元的完全积分

**EN:** Recommended full Gauss integration orders:

| Element | Shape function order $p$ | Full integration | Gauss points |
|---|---|---|---|
| 4-node 2D quadrilateral | $p = 2$ | $2 \times 2$ | 4 points |
| 8-node 2D quadrilateral | $p = 3$ | $3 \times 3$ | 9 points |
| 8-node 3D brick | $p = 2$ | $2 \times 2 \times 2$ | 8 points |
| 20-node 3D brick | $p = 3$ | $3 \times 3 \times 3$ | 27 points |

The requirement comes from: $2n - 1 \geq 2(p - 1)$, where $p$ is the shape function polynomial order and $p - 1$ is the approximate B-matrix polynomial order (after differentiation).

中文： 4节点四边形：2×2 Gauss（4点）；8节点四边形：3×3 Gauss（9点）。完全积分确保被积函数 $\mathbf{B}^T\mathbf{D}\mathbf{B}|\mathbf{J}|$ 中的所有多项式项被精确积分。

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### Q71 🔴 减缩积分(Reduced Integration)

**EN:** Reduced integration uses **fewer** Gauss points than the full integration scheme:

| Element | Full | Reduced |
|---|---|---|
| 4-node quad | $2 \times 2$ | $1 \times 1$ (1 point) |
| 8-node quad | $3 \times 3$ | $2 \times 2$ (4 points) |

**Advantages:** (1) Lower computational cost; (2) Can alleviate **shear locking** and **volumetric locking** in certain problems.

**Disadvantages/Risks:** May introduce **spurious zero-energy modes** — deformation patterns that produce zero strain at all integration points, leading to **hourglass modes**. The element has no stiffness against such deformation.

中文： 减缩积分 = 用比完全积分更少的Gauss点。优点：计算量小、可缓解剪切锁死和体积锁死。缺点：可能引入虚假零能模式（沙漏模式）——单元变形但积分点处应变为零，刚度无法抵抗该变形。

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### Q72 ⭐ 沙漏模式(Hourglass Mode)

**EN:** Hourglass mode (also called zero-energy mode, spurious mode, or kinematic mode) occurs when an element deforms in such a way that **all integration points register zero strain and stress**, yet the element has clearly deformed. The element therefore has **no stiffness** against this deformation mode.

This most commonly occurs with **reduced integration** schemes (e.g., $1 \times 1$ Gauss for a 4-node quadrilateral). With only one integration point at the element center, deformation patterns that preserve the center's shape produce zero strain energy at that point.

**Solutions:** (1) Use full integration; (2) Use hourglass control algorithms; (3) Change element type; (4) Improve mesh quality.

中文： 沙漏模式 = 单元变形了但积分点处应变为零，单元对该变形模式无刚度。最常见于4节点单元的1点减缩积分（只有中心一个积分点）。解决方法：用完全积分、沙漏控制、换单元类型、改进网格质量。

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### Q73 ⭐ 位移 vs 应力精度

**EN:** In displacement-based FEM:

- **Displacements** are **most accurate**, continuous across elements → primary unknowns
- **Strains** are **less accurate** (from displacement derivatives), **discontinuous** across element boundaries
- **Stresses** ($\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$) are similarly **less accurate and discontinuous**
- **Gauss points** give the **most accurate stress/strain** values (superconvergent points)
- Nodal stresses are typically obtained by **extrapolation** from Gauss points or **averaging** across adjacent elements

中文： 精度排序：位移 > 应变 > 应力。位移连续，应力/应变在单元间不连续。Gauss点应力和应变最准（超收敛点）。节点应力需通过外推或平均得到。

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### Q74 🔴 Gauss点处应力最准确的原因

**EN:** Gauss integration points are "superconvergent" points — the error in stress/strain at these points converges at a higher rate than elsewhere in the element. For a 4-node quadrilateral, the $2 \times 2$ Gauss points give stresses that are one order more accurate than at nodes. This is why commercial FEA software typically reports stresses at integration points and then extrapolates to nodes.

中文： Gauss点处的应力具有"超收敛性"——误差递减速率比其他位置快一阶。因此商业FEM软件通常在积分点计算应力，再外推到节点。这也是节点应力需要平均处理的根本原因。

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### Q75 🟠 应力平滑

**EN:** Since stresses are discontinuous across element boundaries, post-processing often involves **stress smoothing** to obtain continuous nodal stress fields:

- **Simple averaging:** $\bar{\boldsymbol{\sigma}}_P = \frac{1}{m}\sum_{e=1}^{m}\boldsymbol{\sigma}_P^{(e)}$ (average of all elements sharing node $P$)
- **Area-weighted averaging:** $\bar{\boldsymbol{\sigma}}_P = \frac{\sum A_e \boldsymbol{\sigma}_P^{(e)}}{\sum A_e}$ (weighted by element areas)

This produces visually smoother stress contours for engineering interpretation.

**中文：** 应力平滑——对共享节点的各单元在节点处的应力做平均（简均或面积加权平均），得到连续的节点应力分布，用于绘制光滑的应力云图。

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### Q76 🟡 2点Gauss积分点位置和权重

**EN:** For 1D Gauss quadrature with $n = 2$:

- Integration points: $x_1 = -1/\sqrt{3} \approx -0.57735$, $x_2 = +1/\sqrt{3} \approx +0.57735$
- Weights: $W_1 = W_2 = 1.0$

For $n = 1$: $x_1 = 0, W_1 = 2$.

For $n = 3$: $x = \{-0.7746, 0, +0.7746\}$, $W = \{5/9, 8/9, 5/9\}$.

In 2D, the points are the tensor product: $(s_i, t_j)$ with weights $W_i W_j$.

**中文：** 2点Gauss：$x = \pm 0.57735$，权重各1.0。3点：$x = \{0, \pm 0.7746\}$，权重 $\{8/9, 5/9, 5/9\}$。二维是张量积：点 $(s_i, t_j)$，权重 $W_i W_j$。

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## L11 — Plate Bending（板弯曲）🔴 样卷未考

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### Q77 🔴 板和梁的区别

**EN:** A **plate** is the 2D extension of a beam. Key differences:

- A beam resists bending about **one axis**; a plate resists bending about **two perpendicular axes** AND **twisting**
- A beam uses 1D theory ($v(x)$); a plate uses 2D theory ($w(x, y)$)
- Beam governing equation: $EI\, d^4v/dx^4 = q$; Plate: $D\nabla^4 w = q$
- A plate is initially **flat**; if initially curved, it is called a **shell**

**中文：** 板是梁的二维推广。梁抵抗单向弯曲，板抵抗两个方向的弯曲+扭转。梁方程 $EI\, w'''' = q$；板方程 $D\nabla^4 w = q$。初始为平面的是板，初始为曲面的是壳。

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### Q78 🔴 Kirchhoff薄板假设

**EN:** The Kirchhoff (thin) plate theory is based on three key assumptions:

1. **Normals remain normal** — lines normal to the mid-surface remain straight and normal after deformation. This implies zero transverse shear strains: $\gamma_{xz} = \gamma_{yz} = 0$.
2. **Normal stress is negligible** — the transverse normal stress is zero: $\sigma_z = 0$. The plate is in a state of plane stress parallel to the mid-surface. Also, thickness change is neglected: $\varepsilon_z = 0$.
3. **Membrane forces are neglected** — in-plane (membrane) displacements of the mid-surface are zero: $u(x, y, 0) = v(x, y, 0) = 0$. This is valid when the transverse deflection is small compared to thickness ($w \ll t$).

**中文：** Kirchhoff薄板假设——(1)法线保持法线：$\gamma_{xz} = \gamma_{yz} = 0$（横向剪切为零）；(2)法向应力忽略：$\sigma_z \approx 0$（平面应力状态），$\varepsilon_z \approx 0$；(3)中面无面内位移：$u_0 = v_0 = 0$（小挠度 $w \ll t$）。这些假设类似于梁的Euler-Bernoulli假设。

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### Q79 🔴 板的弯曲刚度D

**EN:** The bending rigidity (flexural rigidity) of a plate is:

$$\boxed{D = \frac{Et^3}{12(1-\nu^2)}}$$

Analogous to $EI$ for beams. The factor $1/(1-\nu^2)$ accounts for the biaxial stress state in plates. $D$ has units of force × length (e.g., N·m). Note the $t^3$ dependence — doubling the thickness increases bending stiffness by a factor of 8.

中文：板的弯曲刚度 $D = Et^3/[12(1-\nu^2)]$，类似梁的 $EI$。$1/(1-\nu^2)$ 因子来自板的双轴应力状态。$D$ 正比于 $t^3$——厚度翻倍，刚度增加8倍。这是必背公式。

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### Q80 🔴 板的控制方程

**EN:** The governing differential equation for Kirchhoff plate bending:

$$\boxed{D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = q}$$

or compactly: $D\nabla^4 w = q$, where $\nabla^4 = \nabla^2\nabla^2$ is the biharmonic operator.

Compare with beam: $EI\, d^4w/dx^4 = q$. The plate equation is the 2D biharmonic generalization.

中文：板控制方程：$D(w_{xxxx} + 2w_{xxyy} + w_{yyyy}) = q$（双调和方程）。类比梁：$EI\, w'''' = q$。板方程多了混合导数项 $2w_{xxyy}$（来自两个方向的弯曲耦合和扭转）。

---

### Q81 🔴 板位移场

**EN:** Under Kirchhoff assumptions, the 3D displacement components are expressed in terms of the mid-surface deflection $w(x, y)$:

$$u(x,y,z) = -z\frac{\partial w}{\partial x}, \quad v(x,y,z) = -z\frac{\partial w}{\partial y}, \quad w(x,y,z) = w(x,y)$$

The in-plane displacements $u$ and $v$ vary linearly through the thickness and are zero at the mid-surface ($z = 0$). The transverse displacement $w$ is constant through the thickness.

中文：板内位移场：$u = -z\partial w/\partial x, v = -z\partial w/\partial y, w = w(x,y)$。面内位移 $u, v$ 沿厚度线性变化（中面为零），挠度 $w$ 沿厚度不变。这完全类似于梁的 $u = -y\, dv/dx$。

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### Q82 🔴 曲率与应变的关系

**EN:** Plate curvatures are defined as:

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$

The in-plane strains at any distance $z$ from the mid-surface are:

$$\varepsilon_x = z\kappa_x, \quad \varepsilon_y = z\kappa_y, \quad \gamma_{xy} = z\kappa_{xy}$$

The strain varies linearly through the thickness (zero at mid-surface, maximum at surfaces $z = \pm t/2$).

中文：曲率 $\kappa_x = -w_{,xx}, \kappa_y = -w_{,yy}, \kappa_{xy} = -2w_{,xy}$。应变 = z × 曲率，沿厚度线性分布（类似梁的 $\varepsilon = -yv''$）。注意负号约定。

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### Q83 🔴 弯矩-曲率关系

**EN:** The bending and twisting moments per unit length are:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

where $M_x = \int_{-t/2}^{t/2} \sigma_x z\, dz$, $M_y = \int_{-t/2}^{t/2} \sigma_y z\, dz$, $M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z\, dz$.

The moment-curvature material matrix has the same form as the plane stress D-matrix but multiplied by $D$ instead of $E/(1-\nu^2)$.

中文：板的弯矩 = 曲率 × D矩阵（形式同平面应力D矩阵，但因子为 $D = Et^3/[12(1-\nu^2)]$）。$M_x, M_y$ 是单位长度的弯矩，$M_{xy}$ 是单位长度的扭矩。

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## Q84 🔴 板单元每节点自由度

**EN:** A plate bending element has **3 DOFs per node**:

$$\{d_i\} = \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix}$$

where:

- $w_i$ — transverse deflection
- $\theta_{xi} = +\partial w/\partial y$ — rotation about the x-axis
- $\theta_{yi} = -\partial w/\partial x$ — rotation about the y-axis

A 4-node rectangular plate element has **12 DOFs** total ($4 \times 3 = 12$).

中文：板单元每节点3个DOF：挠度 $w$、绕x轴转角 $\theta_x = +\partial w/\partial y$、绕y轴转角 $\theta_y = -\partial w/\partial x$。4节点矩形板单元共12个DOF。注意 $\theta_y$ 的负号。

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## Q85 🔴 板的12项位移多项式

**EN:** For a 4-node rectangular plate element, the transverse displacement is interpolated with a 12-term polynomial:

$$\begin{aligned} w = {} & a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 \\ & + a_7 x^3 + a_8 x^2 y + a_9 x y^2 + a_{10} y^3 \\ & + a_{11} x^3 y + a_{12} x y^3 \end{aligned}$$

The 12 coefficients are determined by the 12 nodal DOFs (4 nodes × 3 DOFs each). Note that the polynomial is not complete — the $x^2 y^2$ term is missing (a complete quartic would have 15 terms). This incompleteness means the element may not satisfy inter-element slope continuity ($C^1$ compatibility) on all edges.

中文：板单元位移多项式含12项（对应12个DOF）。注意缺少 $x^2y^2$ 项——12项多项式不是完全四次（完全四次需15项）。这导致该单元在某些边上不满足$C^1$连续性（斜率不连续），是板单元的经典问题。

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## Q86 🟠 板弯曲的等效节点力

**EN:** For a plate under distributed transverse load $q(x,y)$, the equivalent nodal forces are:

$$\{\mathbf{F}_s\} = \int_A [\mathbf{N}_s]^T q\, dx\, dy$$

where $[\mathbf{N}_s]$ contains the shape functions corresponding to the transverse displacement DOFs. For a uniform load, the equivalent nodal loads include both nodal forces AND nodal moments (analogous to the fixed-end moments in beam elements under uniform load).

中文：板受均布载荷时，等效节点力包含节点力和节点力矩（类比梁的固端力）。通过 $\mathbf{F}_s = \int \mathbf{N}_s^T q\, dA$ 积分得到。

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## Q87 🟠 薄板判据

**EN:** A plate is considered "thin" (Kirchhoff theory applicable) when:

$$\frac{t}{L} < \frac{1}{10} \quad \text{or} \quad t \ll b,\ t \ll c$$

where $t$ is the thickness and $L$ (or $b, c$) is the characteristic in-plane dimension. If the thickness is larger, **transverse shear deformation** becomes significant, and Mindlin-Reissner (thick plate) theory should be used instead (which includes $\gamma_{xz}$ and $\gamma_{yz}$).

**中文：** 薄板条件：$t < L/10$（厚度远小于面内尺寸）。若厚度较大，横向剪切变形不可忽略，需用Mindlin厚板理论（考虑 $\gamma_{xz}, \gamma_{yz}$）。

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### Q88 🟡 板的应力-应变关系

**EN:** Under the Kirchhoff plate assumption ($\sigma_z = 0$), the in-plane stresses are:

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y), \quad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \quad \tau_{xy} = G\gamma_{xy}$$

where $G = E/[2(1+\nu)]$. This is identical to the plane stress constitutive law, applied to each layer through the thickness.

**中文：** 板的面内应力-应变关系 = 平面应力本构（因为 $\sigma_z \approx 0$）。在每个厚度层上应用 $\sigma_x = E/(1-\nu^2)(\varepsilon_x + \nu\varepsilon_y)$ 等。最大应力出现在板的上下表面（$z = \pm t/2$）。

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## L12 — Nonlinearity（非线性）🔴 样卷仅略微涉及

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### Q89 🔴 三类非线性来源

**EN:** The three major sources of nonlinearity in FEM:

1. **Material nonlinearity（材料非线性）**：The stress-strain relationship is not linear — plasticity, hyperelasticity (rubber), creep, damage. The tangent modulus changes with strain and loading history. $\boldsymbol{\sigma} = f(\boldsymbol{\varepsilon}, \text{history})$.
2. **Geometric nonlinearity（几何非线性）**：Large displacements, large rotations, or large strains. The strain-displacement relationship includes quadratic terms (Green-Lagrange strain). Equilibrium must be established on the deformed configuration. Stiffness depends on displacement.
3. **Boundary nonlinearity（边界非线性）**：Contact and impact problems. The contact status (open/closed/sticking/sliding) changes with loading and is unknown a priori. The response is non-smooth.

**中文：** 三大非线性来源——(1)材料非线性：应力-应变关系不是直线（塑性、橡胶超弹性、蠕变、损伤）；(2)几何非线性：大位移/大转动/大应变，应变-位移关系含二次项，平衡方程需写在变形后构形上；(3)边界非线性：接触/碰撞/摩擦，接触状态随加载变化且未知。

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### Q90 🔴 线性 vs 非线性分析的关键区别

**EN:**

| Feature | Linear | Nonlinear |
|---|---|---|
| Governing equation | $\mathbf{Ku} = \mathbf{P}$ | $\mathbf{K}(\mathbf{u}, t)\mathbf{u} = \mathbf{P}(\mathbf{u}, t)$ |
| Stiffness matrix | Constant | Depends on displacement, time, state |
| Scaling of results | **Yes** (proportional) | **No** (cannot simply scale) |
| Superposition | **Valid** | **Not valid** |
| Solution method | Solve once | **Incremental + Iterative** |
| Cost | 1× | 10× to 1000× linear analysis |

**中文：** 线性：K为常数，结果可比例缩放，叠加原理成立，直接求解；非线性：K随位移/时间/状态变化，不可比例缩放，叠加原理不成立，需增量+迭代求解。非线性分析成本 = 几十到上千次线性分析。

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## Q91 🔴 几何非线性的Green-Lagrange应变

**EN:** For large deformation problems, the linear strain $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is insufficient. The **Green-Lagrange strain tensor** includes quadratic terms:

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)}$$

The underlined term $\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}$ is the **nonlinear (quadratic) term** that accounts for large deformation effects. It couples different displacement gradient components.

**中文:** 大变形下，线性应变不够，需用Green-Lagrange应变（含位移梯度的二次项 $\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}$）。这个二次项是大变形分析的核心特征。小变形时二次项可忽略，退化为线性应变。

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## Q92 🔴 接触问题为什么是最困难的非线性问题?

**EN:** Contact problems are among the most difficult nonlinear problems because:

1. **Unknown contact region** — the contacting area is not known a priori and changes during loading
2. **Discontinuous normal velocity** — when contact is established or lost, the normal velocity jumps
3. **Non-smooth friction** — stick-slip transitions cause tangential velocity discontinuities; friction force changes direction abruptly
4. **No-penetration constraint** — the inequality constraint $g_N \leq 0$ (gap cannot be negative) must be enforced
5. **Multiple status changes** — each contact point can switch between open, sticking, and sliding states

**中文:** 接触问题五大难点——(1)接触区域未知且随加载变化；(2)法向速度不连续（接触建立/丧失时跳变）；(3)摩擦非光滑（粘着-滑动转换时切向力突变）；(4)不可穿透约束 $g_N \leq 0$ 需强制满足；(5)每个接触点可在开/粘/滑三态间切换。

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## Q93 🔴 主面(Master) vs 从面(Slave)

**EN:** In contact modeling:

- **Master surface**（主面）: Typically the **stiffer**, **rigid**, or **coarser-meshed** surface
- **Slave surface**（从面）: Typically the **softer**, **deformable**, or **finer-meshed** surface

Rule: Slave nodes cannot penetrate master segments. The slave surface should have a **finer mesh** to accurately resolve the contact pressure distribution. The master should be the surface with the **stiffer** material behavior.

**中文:** 主面 = 刚性/较硬/网格较粗的面；从面 = 较软/网格较细的面。从面节点不能穿透主面。从面网格应更细以准确解析接触压力。主面通常是较硬一侧。

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## Q94 🔴 Coulomb摩擦模型

**EN:** The Coulomb friction model states that the tangential contact traction $t_T$ is bounded by the friction coefficient $\mu$ times the normal contact pressure:

$$\boxed{|t_T| \leq \mu |t_N|}$$

Two states:

- **Sticking**（粘着）: $|t_T| < \mu |t_N|$ — no relative tangential motion
- **Sliding**（滑移）: $|t_T| = \mu |t_N|$ — relative tangential motion occurs; friction force opposes the motion direction

The stick-slip transition makes the tangential response **non-smooth**.

**中文:** Coulomb摩擦：切向力不超过 $\mu$ × 法向力。粘着时 $|t_T| < \mu |t_N|$（无相对滑动）；滑移时 $|t_T| = \mu |t_N|$（摩擦力达极限）。粘着-滑移转换导致切向不光滑。

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### Q95 🔴 Newton-Raphson方法

**EN:** The Newton-Raphson (N-R) method solves nonlinear equations $\mathbf{R}(\mathbf{u}) = \mathbf{P} - \mathbf{F}_{int}(\mathbf{u}) = \mathbf{0}$ iteratively:

$$\boxed{\mathbf{K}_T^{(i)} \Delta \mathbf{u}^{(i)} = \mathbf{R}^{(i)}}$$

$$\boxed{\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \Delta \mathbf{u}^{(i)}}$$

where $\mathbf{K}_T = \partial \mathbf{F}_{int} / \partial \mathbf{u}$ is the **tangent stiffness matrix** and $\mathbf{R}$ is the **residual** (imbalance between external and internal forces). Iteration continues until $\|\mathbf{R}\| < \text{tolerance}$.

**Advantages:** Quadratic convergence near the solution (very fast).

**Disadvantages:** Must recalculate and factorize $\mathbf{K}_T$ each iteration — computationally expensive. May diverge if initial guess is poor.

**中文：** N-R迭代：用切线刚度矩阵 $\mathbf{K}_T$ 求解位移增量，更新位移，直至残差收敛。优点：近解时二次收敛（极快）；缺点：每次迭代需更新切线刚度（计算量大），初值不好可能发散。

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### Q96 🔴 Modified Newton-Raphson vs N-R

**EN:**

| Feature | N-R | Modified N-R |
|---|---|---|
| Tangent stiffness update | **Every iteration** | **Less frequently** (e.g., once per increment) |
| Cost per iteration | High | Lower |
| Convergence rate | Fast (quadratic) | Slower (linear) |
| Total iterations | Fewer | More |
| Best for | Highly nonlinear problems | Mildly nonlinear problems |

The trade-off is between the cost of updating $\mathbf{K}_T$ and the number of iterations needed.

**中文：** N-R每次迭代都更新切线刚度（贵但收敛快）；mN-R少更新（便宜但需更多迭代）。本质是计算成本与收敛速度的权衡。强烈非线性用N-R，弱非线性用mN-R可能更高效。

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### Q97 🔴 增量-迭代求解策略

**EN:** The standard approach for nonlinear FEM analysis is the **incremental-iterative** strategy:

1. **Divide** the total load into increments: $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$
2. **For each increment**, apply the load step $\Delta \mathbf{P}$
3. **Iterate** within the increment using N-R or modified N-R until convergence
4. **Update** state variables (stress, plastic strain, contact status, etc.)
5. **Proceed** to the next load increment

This avoids the divergence that would occur if the full load were applied in one step. Each increment can be thought of as a "small" nonlinear problem.

**中文：** 增量+迭代＝FEM非线性求解的标准策略。将总载荷分段施加，每段内用N-R/mN-R迭代至收敛。这种策略可避免一次性施加全载荷导致的发散。这是所有商业FEM软件处理非线性的基本方法。

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### Q98 🟡 N-R可能发散的情况

**EN:** Newton-Raphson may fail to converge (diverge) when:

1. The tangent stiffness $\mathbf{K}_T$ becomes **zero or negative** (e.g., snap-through buckling, strain softening)
2. The initial guess is too far from the solution
3. The load increment is too large for the degree of nonlinearity

Special techniques: arc-length methods (Riks method) for snap-through; automatic time stepping; line search methods.

中文： N-R在以下情况可能发散：切线刚度变零或变负（跳跃屈曲snap-through、应变软化）、初值太差、载荷增量太大。对策：弧长法(Riks)、自适应步长、线搜索。

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## L13 — Thermal Stress（热应力）🔴 样卷未考

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### Q99 🔴 热应变公式

**EN:** The thermal strain caused by a temperature change $\Delta T$ for an isotropic material is:

$$\boxed{\varepsilon_{ij}^0 = \alpha \Delta T\, \delta_{ij}}$$

where $\alpha$ is the coefficient of thermal expansion (CTE) and $\delta_{ij}$ is the Kronecker delta. In vector form for 3D:

$$\boldsymbol{\varepsilon}^0 = \alpha \Delta T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thermal strain is purely **volumetric** (equal normal strains, no shear strains).

中文： 热应变 $\varepsilon_{ij}^0 = \alpha \Delta T \delta_{ij}$。各向同性材料中，热应变是纯体积应变——各方向正应变相等（$\alpha \Delta T$），剪应变为零。这是必背公式。

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### Q100 🔴 自由热膨胀是否产生应力?

**EN: No.** Free (unconstrained) thermal expansion does NOT produce thermal stress. The body simply expands/contracts without developing internal forces. Thermal stress arises ONLY when:

1. Thermal expansion is **constrained** (e.g., fixed-end bar)
2. There is a **temperature gradient** within the body (differential expansion)
3. The structure is made of materials with **different CTE** ($\alpha$) values

Classic example: A bar fixed at both ends develops compressive stress $\sigma = -E\alpha\Delta T$ when uniformly heated.

中文： 自由热膨胀**不产生**热应力。只有在约束、温度梯度、或材料热膨胀系数差异时，才产生热应力。经典例子：两端固支杆均匀升温 → 压应力 $\sigma = -E\alpha\Delta T$。

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### Q101 🔴 热弹性本构关系

**EN:** With thermal effects, the constitutive relationship becomes:

$$\boxed{\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0)}$$

where $\boldsymbol{\varepsilon}$ is the **total strain** (from both mechanical and thermal effects) and $\boldsymbol{\varepsilon}^0$ is the **thermal strain**. The quantity $\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0 = \boldsymbol{\varepsilon}_m$ is the **mechanical strain** — only this portion produces stress.

中文： 热弹性本构：$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0)$。总应变 $\boldsymbol{\varepsilon}$ 减去热应变 $\boldsymbol{\varepsilon}^0$ 得到机械应变 $\boldsymbol{\varepsilon}_m$，只有机械应变才产生应力。这是区别于纯力学问题的关键。

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### Q102 🔴 热等效节点力

**EN:** The thermal effect enters the FEM equations through the **thermal equivalent nodal force vector**:

$$\boxed{\mathbf{f}_T^e = \int_{V_e} \mathbf{B}^T \mathbf{D} \boldsymbol{\varepsilon}^0 \, dV}$$

The element equation becomes: $\mathbf{k}_e \mathbf{d}_e = \mathbf{f}^e + \mathbf{f}_T^e$

where $\mathbf{f}^e$ is the mechanical load and $\mathbf{f}_T^e$ is the thermal load. After assembly: $\mathbf{Kd} = \mathbf{F} + \mathbf{F}_T$.

中文：热等效节点力 $\mathbf{f}_T^e = \int \mathbf{B}^T \mathbf{D} \boldsymbol{\varepsilon}^0 dV$。热效应被"等效"为一组节点力，与机械力一起装配到右端项。这是热应力FEM的核心公式。

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### Q103 🔴 一维杆热等效节点力

**EN:** For a 1D bar element with constant $E, A, \alpha, \Delta T$:

$$\boxed{\mathbf{f}_T = EA\alpha\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}}$$

These forces act in opposite directions at the two nodes — like a pair of forces trying to expand the element. If the bar is constrained (both ends fixed), these forces produce reactions and internal compressive stress.

中文：一维杆单元热等效节点力 = $EA\alpha\Delta T\{-1, 1\}^T$。两个力大小相等、方向相反，试图"拉长"单元。若杆两端约束，则产生反力和压应力。

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### Q104 🔴 两端固支均匀升温杆的应力

**EN:** For a bar fixed at both ends with uniform temperature increase $\Delta T$:

1. Free thermal strain: $\varepsilon^0 = \alpha\Delta T$
2. Due to fixed ends, total strain: $\varepsilon = 0$
3. Mechanical strain: $\varepsilon_m = \varepsilon - \varepsilon^0 = -\alpha\Delta T$
4. Thermal stress: $\boxed{\sigma = -E\alpha\Delta T}$ (compressive)

End reaction: $F = EA\alpha\Delta T$ (at each support).

The negative sign indicates **compressive** stress — the bar wants to expand but is constrained.

中文：两端固支均匀升温 $\rightarrow \sigma = -E\alpha\Delta T$（压应力）。杆想膨胀但被约束 $\rightarrow$ 产生压应力。端部反力 $F = EA\alpha\Delta T$。

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### Q105 🔴 平面应力 vs 平面应变的热应变

**EN:** The thermal strain vectors differ between plane stress and plane strain:

**Plane stress:** $\boldsymbol{\varepsilon}^0 = \alpha\Delta T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$ (2D vector, $\varepsilon_z$ exists but is not explicitly modeled)

**Plane strain:** $\boldsymbol{\varepsilon}^0 = (1+\nu)\alpha\Delta T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$ (note the $(1+\nu)$ factor!)

The $(1+\nu)$ factor in plane strain arises because the constrained $\varepsilon_z = 0$ condition alters the effective thermal expansion in the modeled plane. This is a common exam trap!

中文：⚠️ 高频易错——平面应力热应变：$\alpha\Delta T\{1,1,0\}^T$；平面应变热应变：$(1+\nu)\alpha\Delta T\{1,1,0\}^T$（多了 $(1+\nu)$ 因子！）。原因是平面应变中 $\varepsilon_z = 0$ 的约束改变了面内的有效热膨胀。

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### Q106 🟠 轴对称热应变

**EN:** For axisymmetric problems, the thermal strain includes the hoop direction:

$$\boldsymbol{\varepsilon}^0 = \alpha\Delta T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} \quad \text{corresponding to } \{\varepsilon_r, \varepsilon_z, \varepsilon_\theta, \gamma_{rz}\}^T$$

All three normal strain components get the same thermal strain $\alpha\Delta T$.

中文：轴对称热应变 = $\alpha\Delta T\{1,1,1,0\}^T$，包含环向分量。三个正应变分量各得 $\alpha\Delta T$。

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### Q107 🟠 热应力题标准解题步骤

**EN:** Standard procedure for thermal stress FEM problems:

1. Determine temperature change $\Delta T$
2. Write thermal strain: $\boldsymbol{\varepsilon}^0 = \alpha\Delta T$
3. Write thermoelastic constitutive law: $\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0)$
4. Write shape functions $\mathbf{N}$ and B-matrix $\mathbf{B}$
5. Compute stiffness: $\mathbf{k}_e = \int \mathbf{B}^T\mathbf{D}\mathbf{B}\,dV$
6. Compute thermal force: $\mathbf{f}_T = \int \mathbf{B}^T\mathbf{D}\boldsymbol{\varepsilon}^0\,dV$
7. Assemble: $\mathbf{Kd} = \mathbf{F} + \mathbf{F}_T$
8. Apply BCs and solve for $\mathbf{d}$
9. Post-process stress: $\boldsymbol{\sigma} = \mathbf{D}(\mathbf{Bd} - \boldsymbol{\varepsilon}^0)$

中文：9步走——定ΔT→写热应变→写本构→写N和B→算刚度→算热力→装配→加BC求解→后处理应力。关键是第6步和第9步，多了热力项和减热应变操作。

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## L14 — Heat Transfer（热传导）🟠 样卷未考

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### Q108 🟠 Fourier热传导定律

**EN:** Fourier's law of heat conduction in 1D:

$$\boxed{q_x = -K_{xx}\frac{dT}{dx}}$$

where $q_x$ is the heat flux (kW/m²), $K_{xx}$ is the thermal conductivity (W/m/K), and $dT/dx$ is the temperature gradient. The negative sign indicates heat flows from hot to cold (down the temperature gradient).

中文：Fourier定律：热流 $q_x$ = −导热系数 × 温度梯度。热量从高温流向低温（负号）。K的单位是W/m/K。这是热传导的基本定律。

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### Q109 🟠 稳态热传导方程

**EN:** Steady-state heat conduction (no time dependence, $dT/dt = 0$):

**1D:** $K_{xx}\frac{d^2T}{dx^2} + Q = 0$

**2D (orthotropic):** $K_{xx}\frac{\partial^2 T}{\partial x^2} + K_{yy}\frac{\partial^2 T}{\partial y^2} + Q = 0$

where $Q$ is the internal heat generation rate per unit volume (kW/m³).

中文：稳态（$dT/dt = 0$）热传导方程——1D：$K_{xx}T'' + Q = 0$；2D：$K_{xx}T_{,xx} + K_{yy}T_{,yy} + Q = 0$。Q是内热源。注意与结构力学平衡方程的类比：$T$（温度）↔ $u$（位移）。

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### Q110 🟠 热传导的FEM基本变量

**EN:** In heat transfer FEM, the **primary unknown is temperature** $T$ (analogous to displacement $u$ in structural FEM). The element equation has the same form:

$$\mathbf{k}_e \mathbf{T}_d = \mathbf{P}_e$$

where $\mathbf{k}_e$ = element conductivity matrix (analogous to stiffness matrix), $\mathbf{T}_d$ = nodal temperature vector, $\mathbf{P}_e$ = element thermal load vector (internal heat generation + surface heat flux).

**中文：** 热传导FEM的基本变量是温度T（类比结构FEM的位移u）。单元方程形式相同：$\mathbf{k}_e \mathbf{T}_d = \mathbf{P}_e$，"刚度矩阵"变成热传导矩阵，"力"变成热载荷。

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### Q111 🟠 热传导B矩阵的作用

**EN:** In heat transfer FEM, the B-matrix maps nodal temperatures to temperature gradients:

$$\begin{Bmatrix} \partial T/\partial x \\ \partial T/\partial y \end{Bmatrix} = \mathbf{B}\mathbf{T}_d$$

This is analogous to the strain-displacement B-matrix in structural FEM: $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}$. The conductivity (stiffness) matrix is $\mathbf{k}_e = \int h\mathbf{B}^T\mathbf{K}\mathbf{B}\,dxdy$, where $\mathbf{K} = \begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix}$ is the conductivity matrix.

**中文：** 热传导的B矩阵将节点温度映射为温度梯度（类比结构FEM的应变-位移B矩阵）。$\mathbf{k}_e = \int h\mathbf{B}^T\mathbf{K}\mathbf{B}\,dxdy$，K是导热系数矩阵。

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### Q112 🟠 热传导边界条件

**EN:** Two types of boundary conditions in heat transfer:

1. **Specified temperature** (Dirichlet): $T = \bar{T}$ on $S_1$ (analogous to displacement BC)
2. **Specified heat flux** (Neumann): $q_n = K_{xx}\frac{\partial T}{\partial x}n_x + K_{yy}\frac{\partial T}{\partial y}n_y$ on $S_2$ (analogous to traction BC)

**中文：** 两类热边界——(1)指定温度（Dirichlet，类比位移BC）；(2)指定热流（Neumann，类比力BC）。热流 = 导热系数 × 温度梯度 · 法向量。

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### Q113 🟠 热传导和结构FEM的类比

**EN:** The FEM procedures for heat transfer and structural analysis are completely analogous:

| Structural | Heat Transfer |
|---|---|
| Displacement $u$ | Temperature $T$ |
| Strain $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}$ | Temperature gradient $\nabla T = \mathbf{B}\mathbf{T}_d$ |
| Stress $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ | Heat flux $\mathbf{q} = -\mathbf{K}\nabla T$ |
| Stiffness $\mathbf{k} = \int \mathbf{B}^T\mathbf{D}\mathbf{B}\,dV$ | Conductivity $\mathbf{k} = \int \mathbf{B}^T\mathbf{K}\mathbf{B}\,dV$ |
| Force $\mathbf{F}$ | Thermal load $\mathbf{P}$ |

The same element types, shape functions, isoparametric formulation, Jacobian, Gauss integration, and assembly procedures are used.

**中文：** 结构和热传导FEM完全类比——位移↔温度、应变↔温度梯度、应力↔热流、刚度矩阵↔传导矩阵。相同的单元类型、形函数、等参变换、Gauss积分都可以直接使用。这是FEM扩展到多物理场的基础。

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## L15a — Vibration & Dynamics（振动与动力学）🟠 样卷未考

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## Q114 🟠 多自由度运动方程

**EN:** The equation of motion for a multi-DOF structural system is:

$$\boxed{\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t)}$$

Physical meaning: **Inertia forces** + **Damping forces** + **Elastic forces** = **Applied forces**

- $\mathbf{M}$ — mass matrix (from kinetic energy)
- $\mathbf{C}$ — damping matrix
- $\mathbf{K}$ — stiffness matrix
- $\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}$ — displacement, velocity, acceleration

**中文：** 动力方程：$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t)$。惯性力+阻尼力+弹性力=外力。M是质量矩阵，C是阻尼矩阵，K是刚度矩阵。这是结构动力学的基本方程。

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## Q115 🟠 无阻尼自由振动

**EN:** For undamped free vibration ($\mathbf{f} = \mathbf{0}, \mathbf{C} = \mathbf{0}$):

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}$$

Assuming harmonic motion $\mathbf{u}(t) = \boldsymbol{\phi}\sin(\omega t)$, we get the **generalized eigenvalue problem**:

$$\boxed{(\mathbf{K} - \omega^2\mathbf{M})\boldsymbol{\phi} = \mathbf{0}}$$

For non-trivial solutions: $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$. This yields $n$ eigenvalues $\omega_i^2$ (squared natural frequencies) and $n$ eigenvectors $\boldsymbol{\phi}_i$ (mode shapes), where $n$ is the number of DOFs.

**中文：** 无阻尼自由振动 → 广义特征值问题 $(\mathbf{K} - \omega^2\mathbf{M})\boldsymbol{\phi} = 0$。n个DOF就有n个固有频率和n个模态。固有频率通过 $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ 求解。

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## Q116 🔴 集中质量矩阵 vs 一致质量矩阵

**EN:** Two types of mass matrices:

**Lumped mass matrix:** Mass is concentrated at nodes → **diagonal matrix**. For a bar element: $\mathbf{M} = \frac{\rho AL}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$. Simple, computationally efficient.

**Consistent mass matrix:** Derived from the same shape functions as the stiffness matrix: $\mathbf{M} = \int_V \rho\mathbf{N}^T\mathbf{N}\,dV$. For a bar element: $\mathbf{M} = \frac{\rho AL}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$. More accurate for a given mesh, but non-diagonal.

The consistent mass matrix follows the same formulation philosophy as the stiffness matrix (hence "consistent").

**中文：** 集中质量矩阵 = 对角矩阵（质量集中在节点），计算效率高；一致质量矩阵 = $\int \rho\mathbf{N}^T\mathbf{N}dV$（与刚度矩阵使用相同的形函数），更准确但非对角。杆单元一致质量矩阵 = $\frac{\rho AL}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$。

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## Q117 🟠 固有频率公式

**EN:** For a single-DOF system, the circular natural frequency is:

$$\omega = \sqrt{\frac{k}{m}} \quad [\text{rad/s}]$$

The cyclic frequency: $f = \frac{\omega}{2\pi}$ [Hz].

For a fixed bar: $\omega_1 = \frac{\pi}{L}\sqrt{\frac{E}{\rho}}$ (first axial natural frequency).

For MDOF systems: $\omega_i = \sqrt{\lambda_i}$ where $\lambda_i$ are eigenvalues of $\mathbf{K}\boldsymbol{\phi} = \lambda\mathbf{M}\boldsymbol{\phi}$.

中文：单DOF固有频率 $\omega = \sqrt{k/m}$（单位rad/s）；频率 $f = \omega/(2\pi)$（Hz）。多DOF系统：$\omega_i$ 通过特征值求解。

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### Q118 🔴 低阶模态比高阶模态更准确

**EN:** In FEM dynamic analysis, **lower modes (smaller $\omega_i$) are more accurate** than higher modes. This is because:

- Lower modes have longer wavelengths, requiring fewer elements per wavelength for accurate representation
- Higher modes have shorter wavelengths, and the FE discretization cannot capture them well unless the mesh is very fine
- A rule of thumb: at least 6-10 elements per wavelength are needed for acceptable accuracy

For a given mesh, only the lowest $n/3$ to $n/2$ modes are typically considered reliable ($n$ = number of DOFs).

中文：低阶模态（小频率）比高阶模态更准确。因为低阶模态波长大，有限元离散容易捕捉；高阶模态波长短，需要极密网格才能准确。经验法则：每个波长至少6-10个单元。给定网格下，通常只有最低的1/3到1/2阶模态可靠。

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### Q119 🔴 刚体模态(Rigid Body Modes)

**EN:** Rigid body modes correspond to $\omega = 0$. These represent motions that produce no strain energy — pure translation or rotation of the unconstrained (or insufficiently constrained) structure. In a free-free vibration analysis:

- A 2D structure has 3 rigid body modes (2 translations + 1 rotation)
- A 3D structure has 6 rigid body modes (3 translations + 3 rotations)

**Practical use:** Rigid body modes can be used to **check the FE model** — unexpected zero-frequency modes indicate mechanisms, missing constraints, or disconnected parts.

中文：刚体模态对应 $\omega = 0$（无应变能的运动）。2D有3个刚体模态（2平动+1转动），3D有6个。可用于检查模型——多余的零频率说明有机构、缺少约束或连接错误。

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### Q120 🔴 模态位移的大小无物理意义

**EN:** In normal mode analysis (eigenvalue analysis), the **magnitude of the eigenvector has NO physical meaning** — only the **shape** (relative proportions between DOFs) matters. Eigenvectors can be arbitrarily scaled. Common normalization: $\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i = 1$ (mass normalization) or set the maximum component to 1.

中文：模态分析中，位移大小无物理意义，只有形状（各DOF的相对比例）有意义。可以任意缩放。常见归一化：质量归一化 $\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i = 1$，或最大分量为1。

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### Q121 🟡 模态正交性

**EN:** Mode shapes satisfy orthogonality with respect to $\mathbf{M}$ and $\mathbf{K}$:

$$\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_j = 0, \quad \boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_j = 0 \quad \text{for } i \neq j$$

Mass-normalized: $\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i = 1$ and $\boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_i = \omega_i^2$.

Orthogonality enables modal superposition — the response can be expressed as a linear combination of mode shapes.

中文：不同模态关于M和K正交。正交性使得模态叠加法成为可能（系统响应 = 各模态贡献的线性组合）。这是模态分析的理论基础。

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### Q122 🟡 阻尼比和Rayleigh阻尼

**EN:** Damping ratio: $\zeta = c/c_c$, where $c_c = 2\sqrt{km} = 2m\omega$ is the critical damping. Structural damping: $0 < \zeta < 0.15$, typically 1–5%. For small damping: $\omega_d \approx \omega$.

**Rayleigh (proportional) damping:** $\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K}$, where $\alpha$ and $\beta$ are determined from: $\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta \omega_i}{2}$. This form preserves modal orthogonality and is convenient for FEM implementation.

中文：阻尼比 $\zeta = c/c_c$，结构阻尼约1-5%，小阻尼下 $\omega_d \approx \omega$。Rayleigh阻尼 $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$ 保持了模态正交性，是FEM中最常用的阻尼模型。

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## L15b — Weighted Residual Method（加权残值法）

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### Q123 🔴 FEM方程推导方法有哪几种?

**EN:** Methods to derive FEM element equations:

1. **Direct equilibrium method** — suitable for simple 1D elements (springs, bars)
2. **Energy methods** — Principle of minimum potential energy (elastic materials), Castigliano's theorem, Principle of virtual work (any material)
3. **Variational method** — find a functional equivalent to the PDE
4. **Methods of weighted residuals (MWR)** — most general, applicable to any PDE-BC system

The energy method and MWR are the most widely used. Galerkin's method (a type of MWR) produces identical results to the variational method for self-adjoint problems.

中文：四种方法——(1)直接平衡法（简单1D单元）；(2)能量法（最小势能原理、虚功原理）；(3)变分法（需泛函）；(4)加权残值法（最通用，适用于任何PDE）。Galerkin法（加权残值法的一种）对自伴随问题等价于变分法。

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### Q124 🔴 强形式 vs 弱形式

**EN: Strong form** — the governing PDE must be satisfied at **EACH material point**. This is a very strict requirement. Exact analytical solutions are often impossible.

**Weak form** — the PDE is satisfied in an **integral (weighted average) sense:** $\int_V \mathbf{W} \cdot \mathbf{R} \, dV = \mathbf{0}$, where $\mathbf{R}$ is the residual (error) and $\mathbf{W}$ is a weight function. This is a "big release" — it allows approximate solutions where the residual at individual points may be non-zero, as long as the weighted integral over the domain vanishes.

FEM is based on the weak form. The weak form also naturally introduces boundary conditions through integration by parts.

中文：强形式 = 每个物质点精确满足PDE（太严格，解析解常不可得）；弱形式 = 在加权积分意义上满足PDE（$\int \mathbf{W} \cdot \mathbf{R} \, dV = 0$），允许各点有残差只要加权积分为零。FEM基于弱形式。分部积分将边界条件自然引入弱形式。

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### Q125 🔴 四种加权残值法

**EN:** Four variants of MWR, differing by the choice of weight function $W$:

| Method | Weight Function $W$ | Description |
|---|---|---|
| **Collocation** | Dirac delta | Set $R = 0$ at specific points (collocation points) |
| **Subdomain** | $W = 1$ on subdomain | Set integral of $R$ over subdomains = 0 |
| **Least Squares** | $W = \partial R / \partial a_i$ | Minimize $\int R^2 dV$ w.r.t. unknown coefficients |
| **Galerkin** | $W_i = N_i$ (shape functions) | Use same functions for trial and weight |

**Galerkin is the most commonly used in FEM** — it produces symmetric matrices and, for self-adjoint problems, yields the same result as the variational method.

中文：四种方法——配点法（设R在特定点为0）、子域法（子域内R积分为0）、最小二乘法（极小化R²积分）、Galerkin法（权函数=形函数）。Galerkin最常用，产生对称矩阵，等价于变分法。

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### Q126 ⭐ Galerkin方法的定义

**EN:** In Galerkin's method, the weight functions are the **same shape functions** used to approximate the solution:

$$\boxed{\int_V N_i \cdot R(\tilde{\mathbf{u}})\, dV = 0, \quad i = 1, 2, \ldots, n}$$

where $R(\tilde{\mathbf{u}})$ is the residual obtained by substituting the approximate solution $\tilde{\mathbf{u}} = \sum N_j \mathbf{d}_j$ into the governing PDE.

This yields $n$ algebraic equations for the $n$ unknown coefficients. For the bar element and beam element, Galerkin's method produces the same stiffness matrix as the energy method.

中文： Galerkin法 = 用形函数做权函数，要求残差的加权积分为零：$\int N_i R\, dV = 0$。得到关于未知系数的n个代数方程。对杆和梁，Galerkin法产生的刚度矩阵与能量法完全一致。

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### Q127 🔴 加权残值法的通用性

**EN:** The weighted residual method is the **most general approach** to deriving FEM equations. Unlike the energy/variational method (which requires a functional — only exists for certain PDE types like elasticity and heat transfer), MWR can be applied to **ANY partial differential equation** with **ANY boundary conditions**, even those without a known variational principle.

This generality is a major advantage of MWR/Galerkin's method and why it is the preferred approach in modern FEM.

中文： 加权残值法是最通用的FEM推导方法。能量法/变分法需要有泛函（仅存在于弹性问题、热传导等特定类型），而MWR适用于任何PDE和边界条件组合。这种通用性是MWR的最大优势。

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### Q128 🟡 分部积分在Galerkin法中的作用

**EN:** Integration by parts in Galerkin's method serves two critical purposes:

1. **Reduces differentiation order** — transfers derivatives from the trial function to the weight function, allowing the use of lower-continuity shape functions (e.g., $C^0$ elements for a 2nd-order PDE)
2. **Introduces boundary conditions naturally** — the boundary terms from integration by parts correspond to natural (Neumann/force) boundary conditions. Essential (Dirichlet/displacement) BCs must still be enforced separately.

Example for bar element: $\int_0^L N_i \frac{d}{dx}(AE\frac{du}{dx})dx = N_i AE\frac{du}{dx}\Big|_0^L - \int_0^L AE\frac{du}{dx}\frac{dN_i}{dx}dx$. The boundary term $N_i AE\frac{du}{dx}$ corresponds to the nodal force.

中文： 分部积分的两大作用——(1)降阶：将对试函数的导数转移到权函数上，降低对形函数连续性的要求；(2)引入边界条件：边界项对应自然（力）边界条件。杆单元例子：边界项 $N_i AE\, du/dx$ 对应节点力。

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### Q129 🟡 Galerkin vs 配点法精度对比

**EN:** Galerkin's method generally provides **better accuracy** than the collocation method for the same number of unknown coefficients. In a standard 1D example ($u'' + u + x = 0, u(0) = u(1) = 0$):

- Collocation (1 point): ~21.7% error
- Galerkin (1 unknown): ~0.2% error

This is because Galerkin uses information from the **entire domain** (through integration), while collocation only enforces the equation at **isolated points**. Galerkin's method "averages" the error in a way that minimizes it globally.

中文： Galerkin精度远高于配点法。原因：Galerkin利用整个域的信息（积分），配点法只在孤立点满足方程。Galerkin通过加权积分"平均"误差，实现全局误差最小化。

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## 附录：跨专题高频判断题汇总

| # | 判断陈述 | T/F | 解释 |
|---|---|---|---|
| J1 | 3节点轴对称单元给出常应变 | **F** | 环向应变 $\varepsilon_\theta = u/r$ 随r变化 |
| J2 | 位移型FEM中位移连续，应力/应变不连续 | **T** | 位移$C^0$连续，导数可不连续 |
| J3 | 自由热膨胀产生热应力 | **F** | 只在约束/梯度/材料差异时产生 |
| J4 | 线性分析结果可比例缩放 | **T** | K为常数，叠加原理成立 |
| J5 | 非线性分析结果可比例缩放 | **F** | K随状态变化 |
| J6 | Gauss积分用非等距点 | **T** | 非等距，n点精确2n-1次 |
| J7 | Newton-Cotes积分用非等距点 | **F** | 等距点 |
| J8 | 对称模型可用于所有分析 | **F** | 振动/屈曲会遗漏反对称模态 |
| J9 | FEM位移解是精确解的上界 | **F** | 位移型FEM偏硬 → 下界 |
| J10 | 三角形形函数一阶导数是常数 | **T** | CST中为常数 |
| J11 | 四边形形函数一阶导数是常数 | **F** | 通过$J^{-1}$变换，一般不是常数 |
| J12 | 减缩积分可能产生沙漏模式 | **T** | 积分点应变=0，无刚度 |
| J13 | 梁的FEM挠度函数是4次多项式 | **F** | 3次（Hermite插值） |
| J14 | 平面应力中 $\sigma_z = 0$ | **T** | 薄板假设 |
| J15 | 平面应变中 $\varepsilon_z = 0$ | **T** | 长厚体假设 |
| J16 | 平面应力中 $\varepsilon_z = 0$ | **F** | $\varepsilon_z \neq 0$（自由收缩） |
| J17 | 平面应变中 $\sigma_z = 0$ | **F** | $\sigma_z \neq 0$（纵向约束） |
| J18 | 杆单元能承受弯矩 | **F** | 只能轴向拉压 |
| J19 | 低阶模态比高阶模态更准确 | **T** | 波长长，离散误差小 |
| J20 | 模态位移大小有物理意义 | **F** | 只有形状有意义 |

*Complete Bilingual Q&A Guide · FEM Course L1–L15*
*129 Q&A items + 20 True/False judgments · ~149 topics total*